Military Technical College
Kobry Elkobbah,
Cairo, Egypt
May 25-27,2010

5th International Conference on Mathematics and Engineering Physics (ICMEP-5)

EM-16

The Linear Programming Problem Under Possibilistic And Probabilistic Uncertainties

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Absract

This paper deals with the possibilistic linear programming problem with exponential distribution function which is converted to a usual mathematical programming problem based on maximizing the possibility measure, then the stochastic linear programming problem with multivariate normal distribution is treated using the probability maximization model. For such problems the stability set of the first kind is defined and characterized. The transformation between the possibilistic linear programming problem with exponential distribution function and the stochastic linear programming problem with multivariate normal distribution is discussed also. Finally, numerical examples is given to illustrate the idea developed in this paper.

1. Introduction

Imposing the uncertainty upon the optimization problems is an interesting research topic. The uncertainty may be interpreted as randomness or fuzziness. The randomness occurring in the optimization problems is categorized as the stochastic optimization problems. Many theoretical works that tackle these problems can be found in the scientific literature. Among them, Birge and Louveaux[1] , Prékopa [2], Stancu-Minasian [3] .
On the other hand, the fuzziness occurring in the optimization problems is treated as the fuzzy optimization problems. The fuzzy optimization problems have also been reported in the literature. For example, Słowinski [4] and Delgado et al. [5] gives the main stream of this topic. Lai and Hwang [6,7] also give an insightful survey.
The fusion of randomness and fuzziness occurring in the optimization problems is even a challenge research topic. The book edited by Słowinski and Teghem [8] gives the comparisons between fuzzy optimization and stochastic optimization for the multiobjective programming problems. Inuiguchi and Ramik [9] also gives a brief review of fuzzy optimization and a comparison with stochastic optimization in portfolio selection problem.
Fuzzy programming approach [9] is useful and efficient to treat a programming problem under uncertainty. While classical and stochastic programming approach may require a lot of cost to obtain the exact coefficient value or distribution, fuzzy programming approach does not. From this fact, fuzzy programming approach will be very advantageous when the coefficients are not known exactly but vaguely specified by human expertise[10].

Inuiguchi and Sakawa [11] treated a fuzzy linear programming with a quadratic membership function. Since a quadratic membership function resembles a multivariate normal distribution, they succeeded to show the equivalence between special models of stochastic linear programming problem and fuzzy linear programming problem.

In this paper as a continuation of Inuiguchi and Sakawa [11],

2. Single-Objective possibilistic linear programming problems.

Possibility theory was initially proposed by Zadeh (1978) [12]. Possibility distributions are built on fuzzy sets. An expression such as `` X is F '', where X is a variable and F is a fuzzy set, can represent two kinds of situation:

• On the one hand, the expression ``X is F'' can appear in a situation where the value of X is really known and we estimate to what degree this value is compatible with label F (which meaning depends of course on the context).

• On the other hand, ``X is F'' can also mean that ``all we know about the value of X is that X is F''. In this case, we do not accurately know the value of X. It corresponds to a situation where information is incomplete (with lack of precision and certainty) and where the values of X can only be ordered according to their degree of plausibility or possibility.

When a fuzzy set is used to represent what is known about the value of a singly-valued variable, the degree related to a value expresses the degree of possibility that this value is the true value of the variable. Fuzzy set F is then seen to be a possibility distribution [12]. which expresses preferences for possible values of poorly-known variable X. Several distinct values can simultaneously have a possibility degree equal to 1. In the case of incomplete information, we can compute to which point information ``X is F'' is $\pi(A)$ strong with an assertion such as ``the value of X is in subset A''. Possibility measure expresses that. If A is a crisp subset, then $\pi(A)$ is defined as the maximum of μ_F on A [13].

2.1 Problem statement

In this section, we treat the following linear programming problem with possibilistic parameters in the objective function:

$$\begin{aligned} &Maximize \quad f(x) = c^t x \\ &subject\ to: \ x \in S \end{aligned} \qquad (1)$$

Where: $S = \left\{ x \in R^n \middle| g_r(x) = \sum_{j=1}^{n} a_{rj} x_j \le b_r,\ r = 1,\dots,m, x_j \ge 0,\ j = 1,\dots,n \right\},$

$c = (c_1,\dots,c_n)^t$ is a possibilistic variable .

.

Considering the imprecise nature of the Decision Maker's judgment, it is natural to assume that the Decision Maker (DM) may have imprecise or fuzzy goal (G) for the objective function in problem (1), with $\mu_G : R \to [0,1]$ is a membership function of the fuzzy goal, let us formulate the possibilistic linear programming problem as a usual mathematical programming problem based on maximizing the possibility measure.

The degree of possibility that the objective function value satify the fuzzy goal are represented as

$$\pi_{\tilde{c}x}\left(\tilde{G}\right) = \sup_{r} \min \left\{\mu_{\tilde{c}x}(r), \mu_{\tilde{G}}(r)\right\} \qquad (2)$$

Where $\mu_{\tilde{c}x}(r), \mu_G(r)$ are membership functions of fuzzy sets $(\tilde{c}x)$ and G

Then Problem (1) is formulated as

$$\begin{aligned} &Maximize \quad \pi_{\tilde{c}x}\left(\tilde{G}\right) \\ &subject\ to: \ x \in S \end{aligned} \qquad (3)$$

OR equivalently:

$$\begin{aligned} &\max \min\left\{\mu_{\tilde{c}x}(r), \mu_{\tilde{G}}(r)\right\} \\ &subject\ to: \ x \in S \end{aligned} \qquad (4)$$

Let $(\tilde{c}x)_h = \left\{r / \mu_{\tilde{c}x}(r) \geq h\right\}, \ \mu_G^*(h) = \inf\left\{r \middle| \mu_G(r) \geq h\right\}$ (5)

Then problem(3) can be transformed as follows:

$$\begin{aligned} \max \quad & h \\ s.t.: \quad & \left(\tilde{c}^{\,t} x\right)_h \geq \mu_G^*(h) \\ & 0 \leq h \leq 1, \\ & x \in S \end{aligned} \qquad (6)$$

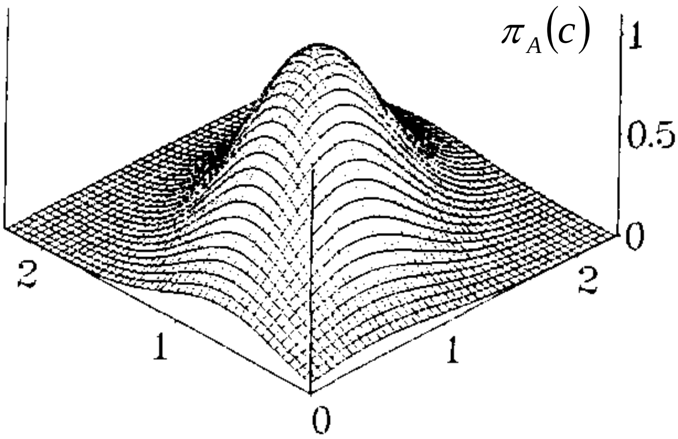
2.2 Exponential membership function

Definition 1[14]: An exponential possibility distribution on an n-dimensional space is represented as

$$\pi_A(c) = \mu_A(c) = \exp\left\{-(c-a)^t D_A^{-1}(c-a)\right\}, \qquad (7)$$

where: A is a label of a possibility distribution, a is a centre vector and D_A is a symmetrical positive definite matrix which is denoted as $D_A > 0$. D_A is corresponding to a covariance matrix in the statistical analysis.

$\pi_A(c)$ can be regarded as a fuzzy vector A that is normal and convex. The parametric representation of (7) is written as $A = (a, D_A)_e$



It is known that the objective function value (cx) is restricted by the following possibility distribution $\pi_{\tilde{c}x}$ defined by the following exponential membership function [ref]:

$$\pi_{\tilde{c}x}(r) = \mu_{\tilde{c}x}(r) = \exp\left\{-\left(r - x^t a\right)^2 \left(x^t D_A x\right)^{-1}\right\}$$

(8)

From (8), $(\tilde{c}^t x)_h$ can be represented as

$$(\tilde{c}^t x)_h = \left[a^t x - \sqrt{(-\ln h) x^t D_A x},\ a^t x + \sqrt{(-\ln h) x^t D_A x}\right]$$

(9)

With the substitution of (9) to Problem (6), we have

$$\begin{aligned} &\textit{Maximize} \quad h \\ &\textit{subject to}: a^t x - \sqrt{(-\ln h) x^t D_A x} \geq \mu_G^*(h), \\ &\qquad\qquad 0 \leq h \leq 1, \\ &\qquad\qquad x \in S \end{aligned}$$

(10)

3. Single-Objective Stochastic linear programming problems.

Stochastic programming is an approach for modeling optimization problems that involve uncertainty. Stochastic programming models try to take advantage of the fact that probability distributions governing those data are known or can be estimated. With René Henrion* we can say that chance constraints offer a way to model reliability in optimization problems.

Stochastic programming, as an optimization method based on the probability theory, have been developing in various ways [Ref], including two stage problem by Dantzig [ref], chance-constrained programming, was pioneered by Charnes and Cooper [Ref] as a means of handling uncertainty by specifying a confidence level at which it is desired that the stochastic constraint holds. After that, Liu [Ref] generalized chance-constrained programming to the case with not only stochastic constraints but also stochastic objectives.

Consider the following stochastic linear programming problem:

$$\begin{aligned} &\textit{Maximize} \quad c^t x \\ &\textit{subject to}: \ x \in S \end{aligned}, \qquad (11)$$

where $c = (c_1, \ldots, c_n)^t$ is a random variable vector obeying a multivariate normal distribution with the mean vector $e = (e_1, \ldots, e_n)^t$ and the covariance matrix V .

The multivariate normal distribution is denoted as $N(e, V)$.

Applying the aspiration criterion model[Ref], the maximization of the objective function in problem (11) will be transformed into the maximization of the probability that each of objective functions is greater than or equal to a certain permissible level z , then problem(11) can be converted as:

$$\begin{aligned} &\textit{Maximize} \quad \Pr\left[c^t x \geq z\right] \\ &\textit{subject to}: \ x \in S \end{aligned} \qquad (12)$$

Applying Katoaka's problem[Ref], then problem(12) is equivalent to the following problem,

$$\begin{aligned} &\textit{Maximize} \quad z \\ &\textit{subject to}: \Pr\left[c^t x \geq z\right] \geq \beta, \\ &\qquad\qquad x \in S \end{aligned} \tag{13}$$

where β is confidence level fixed by the DM

Katoaka's [Ref], consider the predetermined constant $\beta \in [0.5,1]$

Applying the unified model proposed by Geoffrion [Ref] and Ishii et al. [Ref]:

$$\begin{aligned} &\textit{Maximize} \quad (z, \beta) \\ &\textit{subject to}: \Pr\left[c^t x \geq z\right] \geq \beta, \\ &\qquad\qquad \beta \geq 0.5, \\ &\qquad\qquad x \in S \end{aligned} \tag{14}$$

Introducing fuzzy goals Z and P to the objective functions z and β in (14), with $\mu_Z : R \mapsto [0,1]$ and $\mu_P : [0,1] \mapsto [0,1]$ are non-decreasing and upper semi-continuous membership functions of the fuzzy goals Z and P respectively.

Then we have the following mathematical programming model:

$$\begin{aligned} &\textit{Maximize Minimize} \left(\mu_P\left(\Pr\left(c^t x \geq z\right)\right), \mu_Z(z)\right) \\ &\textit{subject to}: x \in S \end{aligned} \tag{15}$$

The above problem can be transformed as follows:

$$\begin{aligned} &\textit{Maximize} \quad h \\ &\textit{subject to}: \ \mu_P\left(\Pr\left(c^t x \geq z\right)\right) \geq h, \\ &\qquad\qquad \mu_Z(z) \geq h, \\ &\qquad\qquad 0 \leq h \leq 1, \\ &\qquad\qquad x \in S \end{aligned} \tag{16}$$

- *How to construct the membership functions* $\mu_Z(z)$ and $\mu_P(\beta)$

Assuming that the optimal solution of (13) be z^1 with $\beta = 0.5$, then the function μ_Z is assumed to satisfy $\mu_Z(z) = 1 \quad \forall \ z \geq z^1$, assuming linear membership function of fuzzy goal z ,

then $\mu_Z(z)$ will be defined as follow:

$$\mu_Z(z) = \begin{cases} 1 & z \geq z^1 \\ \dfrac{(z - z^0)}{(z^1 - z^0)} & z^0 \leq z \leq z^1 \\ 0 & z < z^0 \end{cases} \tag{17}$$

Corresponding to Katoaka's problem[Ref], considering the predetermined constant $\beta \in [0.5,1]$, then the function μ_P is assumed to satisfy linear membership function for simplicity $\mu_P(\beta)=1 \quad \forall\ \beta=1,\ \mu_P(\beta)=0, \forall\ \beta<0.5$, then $\mu_P(\beta)$ will be defined as:

$$\mu_P(\beta)=\begin{cases}1 & \beta=1\\ 2\beta-1 & 0.5\le\beta<1\\ 0 & \beta<0.5\end{cases} \tag{18}$$

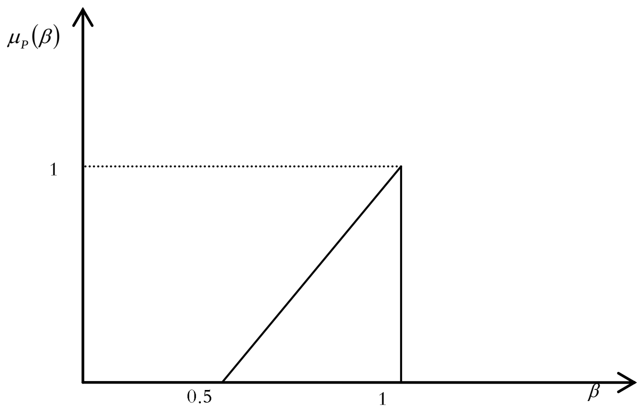


Fig.2. Linear membership function of fuzzy goal P

Thus problem(16) is equivalent to the following problem,

$$\begin{aligned} &\textit{Maximize} \quad h\\ &\textit{subject to}:\ \Pr\left(c^t x \ge \mu_Z^*(h)\right) \ge \mu_P^*(h),\\ &\qquad 0\le h\le 1,\\ &\qquad x\in S \end{aligned} \tag{19}$$

where: $\mu_Z^*(h)=\inf\{r|\mu_Z(r)\ge h\}$, $\mu_P^*(h)=\inf\{r|\mu_P(r)\ge h\}$

Since c obeys $N(e,V)$. Thus, problem (19) can be written as

$$\begin{aligned} &\textit{Maximize} \quad h\\ &\textit{subject to}:\ e^t x-\Phi^{-1}\left(\mu_P^*(h)\right)\sqrt{x^tVx} \ge \mu_Z^*(h),\\ &\qquad 0\le h\le 1,\\ &\qquad x\in S \end{aligned} \tag{20}$$

where Φ is the distribution function of the standard normal distribution, and Φ^{-1} is the inverse function of Φ

3. Transformation between Fuzzy and stochastic Programming Problem.

3.1 Single Objective Linear Programming Problem with Uncertain Parameter in the objective function:

In this section we discuss the transformation between fuzzy linear programming with an exponential membership function and probabilistic linear programming with a multivariate normal distribution.

Based on the above discussion in sections (2) and (3), a possibilistic linear program is equivalent to a stochastic linear program in a special case where problems (10) and (20) are quite similar in their forms.

Assume that $\mu_G^*(\bar{h}) = \mu_Z^*(\bar{h}) \quad \forall \bar{h} \in [0,1]$, with: μ_G and μ_Z are both non-decreasing and upper semi-continuous, then $\mu_G(r) = \mu_Z(r)\ \forall r \in \Re$

and for some $\rho > 0$, these problems are equivalent as follow:

$$a = \rho e,\ D_A = \rho^2 V \qquad (21)$$

$$\sqrt{(-\ln \bar{h})} = \Phi^{-1}(\mu_P^*(\bar{h})) \qquad (22)$$

Let $\sqrt{(-\ln \bar{h})} = k \quad \Rightarrow \bar{h} = \exp\{-k^2\} \qquad \forall k \geq 0$, and for (23)

$$\Phi^{-1}(\mu_P^*(\bar{h})) = k \quad \forall k \geq 0 \Rightarrow \mu_P^*(\bar{h}) = \Phi(k) \Rightarrow \bar{h} = \mu_P(\Phi(k)) \qquad (24)$$

Equations (23) and (24) yield:

$$\exp\{-k^2\} = \mu_P(\Phi(k)) \qquad \forall k \geq 0 \qquad (25)$$

As a result. Solving a possibilistic linear programming problem of the exponential possibility distributed variable is equivalent to solving a stochastic linear programming problem with normal distributed variable with the following parameters:

$a = e,\ D_A = V\ with\ \rho = 1$, and $\mu_G(r) = \mu_Z(r)\ \forall r \in \Re$

$$\mu_P(\beta) = \begin{cases} 1 & \beta = 1 \\ \exp\{-(\Phi^{-1}(\beta))^2\} & 0.5 \leq \beta < 1 \\ 0 & \beta < 0.5 \end{cases} \qquad (26)$$

3.1.1 Illustrative Examples.

ExampleI.

Consider the following possibilistic linear programming problem:

$$\begin{aligned} &Max \quad \hat{z}_1 = \tilde{c}x \\ &s.t. \\ &\qquad 3x_1 + x_2 \leq 12\ , \\ &\qquad 0 \leq x_1 \leq 4, \quad 1 \leq x_2 \leq 5 \end{aligned} \qquad (27)$$

Where: c obeys exponential possibility distributed variable with $(a, D_A)_e$, $a = (2,3)^t$ and

$$D_A = \begin{pmatrix} 4 & 0.5 \\ 0.5 & 2 \end{pmatrix}$$

Assume that a fuzzy goal $\tilde{G}$ is given as: $(9,11,13,18)$

The possibilistic linear programming problem (27) is equivalent to a probabilistic linear programming problem with the following parameters:

$e=(2,3)^t$, $V=\begin{pmatrix}4 & 0.5\\ 0.5 & 2\end{pmatrix}$, $\mu_{\tilde{G}}(r)=\mu_{\tilde{Z}}(r)$,

with $\mu_P(\beta)=\exp\left\{-\left(\Phi^{-1}(\beta)\right)^2\right\}$ $\quad 0.5\leq\beta<1$

Solving problem (13) . Then the optimal solution $(x_1,x_2,z,)=(2.3333,\ 5.0000,\ 14.4698\ )$

Thus, both problems corresponding to(1) and (11) have the same optimal solution:

$(\bar{x}_1,\bar{x}_2,\bar{h})=(2.3333\quad 5.0000\quad 0.7235\ )$

Example II.

Consider the following stochastic linear programming problem:

$$
\begin{aligned}
&Max\ \ \hat{z}_1=\tilde{c}x\\
&s.t.\\
&\qquad 3x_1+x_2\leq 12\ ;\\
&\qquad 0\leq x_1\leq 4,\quad 1\leq x_2\leq 5.
\end{aligned}
\tag{28}
$$

where: c a random variable obeys a multivariate normal distribution with $e=(4,1)^t$ and $V=\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$

Solving problem (13) , with $\beta=0.5$. Then the optimal solution $(x_1,x_2,z,)=(3.6667\ \ ,1.0000\ \ ,\ 15.6667\ \ )$

Corresponding to problem(17), let $z^0=0$, then: $\mu_Z(r)=\dfrac{r}{15.6667}\qquad \forall r>0$.

And according to problem(18): $\mu_P(\beta)=2\beta-1$

The equivalent possibilistic linear programming problem has the following parameters:

$a=(4,1)^t$, $D_A=\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$, $\mu_G(r)=\mu_Z(r)=\dfrac{r}{15.6667}$,

and from equation(26), $\exp\left\{-\beta^2\right\}=\mu_P(\Phi(\beta))=2\Phi(\beta)-1$

Thus, both problems corresponding to(1) and (11) have the same optimal solution:

$(\bar{x}_1,\bar{x}_2,\bar{h})=(3.6667,1.0\ \ 000,\ 0.8136\ \ )$

3.2 Single Objective Nonlinear Programming Problem with Uncertain Parameter:

3.2.1 A stochastic nonlinear programming problem with random parameter in the objective function can be stated in the following form:

$$
\begin{aligned}
&Minimize\quad f(x)=\bar{c}^{\,t}x\\
&subject\ to:\ x\in S=\left\{x\in R^n\,\middle|\,g_r(x)\leq 0,\ r=1,\ldots,m,\quad x_{L_j}\leq x_j\leq x_{U_j},\ \ j=1,\ldots,n\right\}
\end{aligned}
$$

(29)

Where: x_{L_j},x_{U_j} are lower and upper bounds of the j_{th} variable respectively.

In the above problem the objective function of the problem depend on a vector of continuous random variable. $c=(c_1,\ldots,c_n)^t$ is a random variable with any distribution function(normal or uniform or gamma ,..extra).

1. Apply the expected value criterion [Ref], to the stochastic objective function of the problem(29) and we obtain the equivalent deterministic problem (E) as follow:

$$(E)\quad \begin{array}{l} Minimize \quad \bar{f}(x)=E(\bar{c}^t x) \\ subject\ to: x\in S=\left\{x\in R^n \middle| g_r(x)\le 0,\ r=1,\ldots,m,\ \ x_{L_j}\le x_j\le x_{U_j},\ \ j=1,\ldots,n\right\} \end{array}$$

(30)

Where $\bar{c}$ is the expected value of the random variable. Thus the optimal solution x^* of the stochastic nonlinear programming problem stated in Eq. (29) can be obtained by solving an equivalent deterministic nonlinear programming problem (30) by using any available nonlinear programming package .

2. Let us now consider applying Kataoka's criterion[Ref] to the stochastic problem(29), we must fix a value f (aspiration level of the problem's objective function)**,** we obtain the equivalent deterministic problem $K(\beta)$as follow:

$$K(\beta)\quad \begin{array}{l} Minimize \quad f \\ subject\ to: \Pr\left[\bar{c}^t x\ge f\right]\ge\beta, \\ \qquad x\in S=\left\{x\in R^n \middle| g_r(x)\le 0,\ r=1,\ldots,m,\ \ x_{L_j}\le x_j\le x_{U_j},\ \ j=1,\ldots,n\right\} \end{array}$$

(31)

Case of normal distribution.

Let $c=(c_1,\ldots,c_n)^t$ is a random variable vector obeying a multivariate normal distribution with the mean vector $e=(e_1,\ldots,e_n)^t$ and the covariance matrix V, then problem (31) can be written as:

$$\begin{array}{l} Minimize \quad f \\ subject\ to: e^t x+\Phi^{-1}(\beta)\sqrt{x^t V x}\le f, \\ \qquad g_r(x)\le 0,\ r=1,\ldots,m, \\ \qquad x_{L_j}\le x_j\le x_{U_j},\ \ j=1,\ldots,n \end{array}$$

(32)

where Φ is the distribution function of the standard normal distribution, and β is the predetermined confidence level.

3.2.2 A fuzzy nonlinear programming problem with fuzzy parameters in the objective function and constraint:

Consider the following fuzzy nonlinear programming problem:

$Minimize \quad E(\bar{c}^t)\tilde{x}$

$subject\ to: \tilde{x} \in S = \{\tilde{x} \in R^n | g_r(\tilde{x}) \le 0,\ r = 1,...,m,\ \ \tilde{x}_j \ge 0,\ \ j = 1,...,n\}$

(33)

In the fuzzy problem (33) $\tilde{x}_j$ is the vector of fuzzy numbers whose membership functions are $\mu_{\tilde{x}_j}(x_j)$, $j = 1,\ldots,n$ involved in the objective function ($\bar{c}^t\tilde{x}$) and in the constraint functions $g_r(\tilde{x}),\ r = 1,..., m$

• Based on the definition of $\alpha -$level set or $\alpha -$cut [Ref], of the fuzzy numbers $\tilde{x}_j$, $j = 1,\ldots,n$ and for a certain degree $\alpha \in [0,1]$, problem (33) can be understood as the following non fuzzy $\alpha -$single objective nonlinear programming problem as follow:

$Minimize \quad E(\bar{c}^t)x$

$subject\ to: g_r(x) \le 0,\ r = 1,\ldots,m,$

$x_j \in L_\alpha(\tilde{x}_j)$

$x_j \ge 0,\ \ j = 1,\ldots,n$

(34)

The constraint $x_j \in L_\alpha(\tilde{x}_j)$ is equivalent to $x_{L_j} \le x_j \le x_{U_j}$, $j = 1,\ldots,n$ provided that x_{L_j} and x_{U_j} are lower and upper bounds respectively of the variables x_j .

So problem (34) can be written in the following form:

$Minimize \quad E(\bar{c}^t)x$

$subject\ to: g_r(x) \le 0,\ r = 1,\ldots,m,$

$x_{L_j} \le x_j \le x_{U_j}$, $j = 1,\ldots,n$

$x_j \ge 0.$

(35)

The optimal solution x^* of the above deterministic nonlinear programming problem can be found easily by using any available nonlinear programming package.

Results:

Comparing problem(30) with problem(35), one can realize that these two problems are the same.

• This means that the optimal solution of problem(29) can be found by solving the equivalent deterministic version (35) .

• The parameter are random variable in the objective function only in the stochastic problem.

• The parameter are fuzzy parameters in the objective function and in the constraint.

3.2.3 A fuzz nonlinear programming problem with fuzzy parameters in the constraint can be stated in the following form:

$Minimize \quad f$

$subject\ to: e^t\tilde{x} + \Phi^{-1}(\beta)\sqrt{\tilde{x}^t V\tilde{x}} \le f,$

$g_r(\tilde{x}) \le 0,\ r = 1,\ldots,m,$

$\tilde{x}_j \ge 0,\ \ j = 1,\ldots,n$

(36)

In the above fuzzy problem(36), is a vector of fuzzy parameters $\tilde{x}_j,\ \ j = 1,..., n$ involved in the constraint functions.

Now, we assume that $\tilde{x}_j$, $j = 1,...,n$ are fuzzy numbers whose membership functions are $\mu_{\tilde{x}_j}(x_j)$, $j = 1,\ldots,n$.

Based on the definition of $\alpha -$cut , with $x_j \in L_\alpha(\tilde{x}_j)$ problem (36) can be written as the following non fuzzy problem:

$$
\begin{aligned}
& Minimize \quad f \\
& subject\ to: e^t x + \Phi^{-1}(\beta)\sqrt{x^t V x} \le f, \\
& \qquad g_r(x) \le 0,\ r = 1,\ldots,m, \\
& \qquad x_{L_j} \le x_j \le x_{U_j},\ \ j = 1,\ldots,n
\end{aligned}
\tag{37}
$$

It should be noted that the constraints $x_j \in L_\alpha(\tilde{x}_j)$, $j = 1,\ldots,n$ have been replaced by the equivalent constraint $x_{L_j} \le x_j \le x_{U_j}$, $j = 1,\ldots,n$. Provided that x_{L_j}, x_{U_j} are lower and upper bounds of the variable x_j respectively.

3.2.4 An illustrative example:

Consider the following stochastic nonlinear programming problem with random parameters in the objective function:

$$
\begin{aligned}
& Min \quad N(1,1)x_1^2 + N(1,1)x_2^2 , \\
& s.t \quad x_1^2 + x_2^2 \le 25 , \\
& \qquad 3x_1 + x_2 \le 12 , \\
& \qquad 0.5 \le x_1 \le 4, \quad 1 \le x_2 \le 5.
\end{aligned}
\tag{38}
$$

According to problem (32) the equivalent deterministic of problem (38) can be written as:

$$
\begin{aligned}
& Minimize \quad f \\
& subject\ to: x_1^2 + x_2^2 + 1.645\sqrt{x_1^4 + x_2^4} \le f, \\
& \qquad x_1^2 + x_2^2 \le 25 , \\
& \qquad 3x_1 + x_2 \le 12 , \\
& \qquad 0.5 \le x_1 \le 4, \quad 1 \le x_2 \le 5.
\end{aligned}
\tag{39}
$$

Let $\beta = 0.95$, $\Phi^{-1}(0.95) = 1.645$, $x^* = [0.5,1.0]$, $f = 2.9456$

On the other hand, a nonlinear programming problem having fuzzy parameters in the constraints can be formulated as follows:

$$
\begin{aligned}
& Minimize \quad f \\
& subject\ to: \tilde{x}_1^2 + \tilde{x}_2^2 + 1.645\sqrt{\tilde{x}_1^4 + \tilde{x}_2^4} \le f, \\
& \qquad \tilde{x}_1^2 + \tilde{x}_2^2 \le 25 , \\
& \qquad 3\tilde{x}_1 + \tilde{x}_2 \le 12 , \\
& \qquad 0 \le \tilde{x}_1, \quad \tilde{x}_2 \ge 0
\end{aligned}
\tag{40}
$$

The nonfuzzy $\alpha-$cut nonlinear programming problem equivalent to problem(40) can be written as:

$$\begin{aligned} &Minimize \quad f \\ &subject\ to: x_1^2+x_2^2+1.645\sqrt{x_1^4+x_2^4} \le f, \\ &\qquad x_1^2+x_2^2 \le 25, \\ &\qquad 3x_1+x_2 \le 12, \\ &\qquad x_1 \in L_\alpha(\tilde{x}_1), \\ &\qquad x_2 \in L_\alpha(\tilde{x}_2) \end{aligned} \tag{41}$$

Where $L_\alpha(\tilde{x}_1)$, $L_\alpha(\tilde{x}_2)$: are defined as the $\alpha-$level set of the fuzzy numbers x_1 , x_2 respectively.

***Problem (41) is equivalent to problem (39) provided that*:**

$L_\alpha(\tilde{x}_1)=\ 0.5 \le x_1 \le 4$,

$L_\alpha(\tilde{x}_2)= 1 \le x_2 \le 5.$

Then the fuzzy parameters in problem (40) are characterized by the following fuzzy numbers:

$\tilde{x}_1=(0,1,3,5)$ **,** $\tilde{x}_2=(0,2,4,6)$

and we assume that the membership function corresponding to the fuzzy numbers $x_j,\ j=1,2$ takes the form:

$$\mu_{\tilde{x}_j}(x_j)=\begin{cases} 0 & x_j \le r_1, \\ 1-\left(\dfrac{x_j-r_2}{r_1-r_2}\right)^2 & r_1 \le x_j \le r_2, \\ 1 & r_2 \le x_j \le r_3, \\ 1-\left(\dfrac{x_j-r_3}{r_4-r_3}\right)^2 & r_3 \le x_j \le r_4, \\ 0 & x_j \ge r_4. \end{cases}$$

(42)

with $\alpha=0.75$ and the constraints $x_1 \in L_\alpha(\tilde{x}_1)$ and $x_2 \in L_\alpha(\tilde{x}_2)$ have been replaced by the equivalent constraints $0.5 \le x_1 \le 4$ and $1 \le x_2 \le 5$. **the problem (41) is the same as problem (39), and the optimal solution for both problem is:**

$x^*=[0.5,1.0],\quad f=2.9456$

4. Multi Objective linear Programming Problem with Uncertain Parameter:

4.1 Stochastic multiobjective programming problem with random parameter in the objective functions.

Consider the following multiobjective linear programming problem with random variable coefficient stated in the objective functions:

$$Minimize \quad \bar{z}_i(x,\bar{c}) = \bar{c}_i x \ , \qquad i = 1,\ldots,k$$
$$subject\ to: \ x \in S = \left\{ x \in R^n \middle| \begin{array}{l} g_r(x) \le 0, \ r = 1,\ldots,m, \\ x_j \ge 0, \quad j = 1,\ldots,n \end{array} \right\} \qquad (43)$$

In (43), x is an n-dimensional decision variable column vector. The coefficients $\bar{c}_{ij}$, $j = 1,..., n$

of the vector $\bar{c}_i$ are random variables obeying a multivariate normal distribution with the mean vector e and the covariance matrix V

We will now deal with the application of maximum probability to stochastic multiobjective programming problem(SMP) (43).

In this case, the decision maker (DM) must fix *a priori* an aspiration level, u_i , $i = 1,..., k$ for each stochastic objective function and find the vector x, in which the probability of the i_{th} objective function not being greater than the aspiration level fixed is maximum: $P(\bar{z}_i(x,\bar{c}) \le u_i)$.

Then problem(43) is equivalent to the following problem,

$$Maximize \quad \Pr(\bar{z}_i(x,\bar{c}) \le u_i) \ , \ i = 1,\ldots,k$$
$$s.t. \quad x \in S = \left\{ x \in R^n \middle| \begin{array}{l} g_r(x) \le 0, \ r = 1,\ldots,m, \\ x_j \ge 0, \quad j = 1,\ldots,n \end{array} \right\}$$

(44)

Let us now consider solving the weighted problem by apply Kataoka's criterion. The resulting problem, for a probability β is:

$$Minimize \quad \sum_{i=1}^{k} w_i e_i x + \Phi^{-1}(\beta) \sqrt{\sum_{i=1}^{k} w_i^2 x^t V_i x + 2\sum_{i,s=1}^{k} w_i w_s x^t V_{is} x},$$
$$subject\ to: \ x \in S = \left\{ x \in R^n \middle| g_r(x) \le 0, \ r = 1,\ldots,m, \ \ x_{L_j} \le x_j \le x_{U_j} , \ \ j = 1,\ldots,n \right\}$$

(45)

Where $w = (w_1,..., w_k)$ is the weight vector, $w \in \Re$

The optimal solution x^* of the above deterministic programming problem(45) is an efficient solution to problem(43).

Introducing fuzzy goals Z and P in the objective functions for u_i and β_i in (44), with $\mu_Z : R \mapsto [0,1]$ and $\mu_P : [0,1] \mapsto [0,1]$ are non-decreasing and upper semi-continuous membership functions of the fuzzy goals Z and P respectively.

Then we have the following mathematical programming model:

$$Maximize\ Minimize\left(\mu_{P_i}(\Pr(\bar{z}_i(x,\bar{c}) \le u_i)), \mu_{Z_i}(u_i)\right), \quad i = 1,\ldots,k$$
$$subject\ to: x \in S$$

(46)

The above problem can be transformed as follows:

$$
\begin{aligned}
&Maximize \quad h \\
&subject\ to: \ \mu_{P_i}\left(\Pr\left(\bar{z}_i(x,\bar{c}) \le u_i\right)\right) \ge h, \\
&\qquad \mu_{Z_i}(u_i) \ge h, \\
&\qquad 0 \le h \le 1, \\
&\qquad x \in S
\end{aligned}
$$

(47)

Thus problem (46) is equivalent to the following problem,

$$
\begin{aligned}
&Maximize \quad h \\
&subject\ to: \ \Pr\left(c_i^{\,t} x \le \mu_{Z_i}^*(h)\right) \ge \mu_{P_i}^*(h), \\
&\qquad 0 \le h \le 1, \\
&\qquad x \in S
\end{aligned}
$$

(48)

where: $\mu_{Z_i}^*(h) = \inf\{r \mid \mu_{Z_i}(r) \ge h\},\ \mu_{P_i}^*(h) = \inf\{r \mid \mu_{P_i}(r) \ge h\},\ i = 1,\ldots,k$

Since c obeys $N(e,V)$. Thus, problem (48) can be written as

$$
\begin{aligned}
&Maximize \quad h \\
&subject\ to: e_i^{\,t} x + \Phi^{-1}\left(\mu_{P_i}^*(h)\right)\sqrt{x^t V_i x} \le \mu_{Z_i}^*(h), \\
&\qquad 0 \le h \le 1, \\
&\qquad x \in S
\end{aligned}
$$

(49)

4.2. Multi-Objective possibilistic linear programming problems with exponential membership function.

Consider the following multiobjective linear programming problem with fuzzy variable coefficient with exponential membership function stated in the objective functions:

$$
\begin{aligned}
&Minimize \quad \tilde{z}_i(x,\tilde{c}) = \tilde{c}_i x, \qquad i = 1,\ldots,k \\
&subject\ to: \ x \in S = \left\{ x \in R^n \,\middle|\, \begin{matrix} g_r(x) \le 0, & r = 1,\ldots,m, \\ x_j \ge 0, & j = 1,\ldots,n \end{matrix} \right\}
\end{aligned}
$$

(50)

$\tilde{c}$ is fuzzy variable with exponential membership function $A = (a, D_A)_e$ in each objective function.

After determining the membership functions for each of the objective functions and adopting the fuzzy decision of Bellman and Zadeh (1970) [Ref], the resulting problem to be solved is:

$$
\underset{x \in S}{Maximize} \quad \left\{ \min_{i=1,\ldots,k} \left(\mu_i\left(\tilde{z}_i(x,\tilde{c})\right)\right)\right\} \tag{51}
$$

OR equivalently:

$$\begin{aligned} \max \quad & h \\ s.t.: \quad & \left(\tilde{c}_i^t x\right)_h \geq \mu_{Gi}^*(h) \\ & 0 \leq h \leq 1, \\ & x \in S \end{aligned} \tag{52}$$

With the substitution of (9) to Problem (52), we have

$$\begin{aligned} Maximize \quad & h \\ subject\ to: \ & a_i{}^t x + \sqrt{(-\ln h) x^t D_{A_i} x} \leq \mu_{G_i}^*(h), \\ & 0 \leq h \leq 1, \\ & x \in S \end{aligned}$$

(53)

Since problems (53) and (49) are quite similar in their forms.

Assume that $\mu_G^*(\bar{h}) = \mu_Z^*(\bar{h}) \quad \forall \bar{h} \in [0,1]$, with: μ_G and μ_Z are both non-decreasing and upper semi-continuous, then $\mu_G(r) = \mu_Z(r) \ \forall r \in \Re$, and for some $\rho > 0$, these problems are equivalent as follow:

$$a = \rho e, D_A = \rho^2 V, \quad \sqrt{(-\ln h)} = \Phi^{-1}\left(\mu_P^*(h)\right)$$

(54)

As a result, solving a possibilistic multiobjective linear programming problem of the exponential possibility distributed variable is equivalent to solving a stochastic multiobjective linear programming problem with normal distributed variable with the following parameters: $a = e, \ D_A = V \ with \ \rho = 1$, and $\mu_G(r) = \mu_Z(r) \ \forall r \in \Re$

4.3 An illustrative Example:

Let us consider the following stochastic bi-objective programming problem:

$$\begin{aligned} Minimize \quad & f_1 = \bar{c}_{11} x_1 + \bar{c}_{12} x_2 \\ Minimize \quad & f_2 = \bar{c}_{21} x_1 + \bar{c}_{22} x_2 \\ s.t. \quad & x_1 + 2x_2 \geq 4 \\ & 0.2 \leq x_1 \leq 4.8 \\ & 0.4 \leq x_2 \leq 3.8 \end{aligned} \tag{55}$$

Where $\bar{c} = (\bar{c}_{11}, \bar{c}_{12}, \bar{c}_{21}, \bar{c}_{22})$ being a random vector with multivariate normal distribution with mean values $e = (0.5, 1, 1, 2.5)^t$ and with positive definite covariance matrix:

$$V = \begin{pmatrix} 25 & 0 & 0 & 3 \\ 0 & 25 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 3 & 0 & 0 & 9 \end{pmatrix}$$

According to problem (45), problem (55) is converted to the following deterministic one:

$$Minimize \quad w_1(0.5x_1 + x_2) + w_2(x_1 + 2.5x_2) + \Phi^{-1}(\beta)\sqrt{w_1^2(25x_1^2 + 25x_2^2) + w_2^2(x_1^2 + 9x_2^2) + 12w_1w_2x_1x_2}$$

$$subject\ to: \ x_1 + 2x_2 \geq 4$$

$$0.2 \leq x_1 \leq 4.8$$

$$0.4 \leq x_2 \leq 3.8$$

Let $\beta =$ 0.95, $\Phi^{-1}(0.95) = 1.645$, $w = (0.2, 0.8)^t$ we obtain the solution

$x^* = [2.26, 0.87]$

Suppose fuzzy goals Z and P are defined by:

$$\mu_Z(r) = \max(0, \min(0.03r, 1)) \qquad \forall r > 0$$

$$\mu_P(r) = \begin{cases} 1 & if \ r = 1 \\ 2r - 1 & if \ \ 0.5 \leq r < 1 \\ 0 & if \ \ r < 0.5 \end{cases}$$

The equivalent possibilistic multiobjective linear programming problem has the following parameters:

For the first objective $f_1:\ a_1 = (0.5, 1)^t$, $D_{A1} = \begin{pmatrix} 25 & 3 \\ 3 & 25 \end{pmatrix}$

For the second objective $f_2:\ a_2 = (1, 2.5)^t$, $D_{A1} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$

$$\mu_G(r) = \max(0, \min(0.03r, 1)), \qquad \forall r > 0$$

$$\exp\{-r^2\} = \mu_P(\Phi(r)), \qquad \forall r > 0$$

Indeed, both problems have the same optimal solution $[x_1, x_2, h] = [2.26, 0.87, 0.77]$

5. Multi Objective nonlinear Programming Problem with Uncertain Parameter:

5.1 A stochastic multiobjective nonlinear programming problem with random parameter in the objective functions.

Consider the following multiobjective nonlinear programming problem with random variable coefficient stated in the objective functions :

$$Minimize \quad \bar{c}_i x \ , \qquad i = 1, \ldots, k$$

$$subject\ to: \ x \in S = \left\{ x \in R^n \middle| g_r(x) \leq 0, \ r = 1, \ldots, m, \ \ x_{L_j} \leq x_j \leq x_{U_j}, \ \ j = 1, \ldots, n \right\}$$

(56)

In (43), x is an n-dimensional decision variable column vector. The coefficients $\bar{c}_{ij}$, $j = 1, ..., n$

of the vector $\bar{c}_i$ are random variables obeying a multivariate normal distribution with the mean vector e and the covariance matrix V

Let us now consider solving the weighted problem by apply Kataoka's criterion. The resulting problem, for a probability β is:

$$Minimize \quad \sum_{i=1}^{k} w_i e_i x + \Phi^{-1}(\beta)\sqrt{\sum_{i=1}^{k} w_i^2 x^t V_i x + 2\sum_{i,s=1}^{k} w_i w_s x^t V_{is} x},$$

$$subject\ to: \ x \in S = \left\{ x \in R^n \middle| g_r(x) \leq 0, \ r = 1, \ldots, m, \ \ x_{L_j} \leq x_j \leq x_{U_j}, \ \ j = 1, \ldots, n \right\}$$

(57)

Where $w = (w_1, ..., w_k)$ is the weight vector, $w \in \Re$

The optimal solution x^* of the above deterministic programming problem(44) is an efficient solution to problem(43).

5.2 A fuzzy multiobjective nonlinear programming problem with fuzzy parameter in the objective functions and constraints.

Consider the following multiobjective fuzzy nonlinear programming problem:

$$Minimize \quad \sum_{i=1}^{k} w_i e_i \tilde{x} + \Phi^{-1}(\beta)\sqrt{\sum_{i=1}^{k} w_i^2 \tilde{x}^t V_i \tilde{x} + 2\sum_{i,s=1}^{k} w_i w_s \tilde{x}^t V_{is} \tilde{x}},$$

$$subject\ to: \ \tilde{x} \in S = \left\{\tilde{x} \in R^n \middle| g_r(\tilde{x}) \le 0,\ r = 1,\ldots,m,\ \ \tilde{x}_j \ge 0,\ \ j = 1,\ldots,n\right\}$$

(58)

In the fuzzy problem (45) $\tilde{x}_j$ is the vector of fuzzy numbers whose membership functions are $\mu_{\tilde{x}_j}(x_j)$, $j = 1,\ldots,n$ involved in the objective function and in the constraint functions.

- Based on the definition of $\alpha -$level set , of the fuzzy numbers $\tilde{x}_j$, $j = 1,\ldots,n$ and for a certain degree $\alpha \in [0,1]$, problem (45) can be understood as the following non fuzzy $\alpha -$multi objective programming problem as follow:

$$Minimize \quad \sum_{i=1}^{k} w_i e_i \tilde{x} + \Phi^{-1}(\beta)\sqrt{\sum_{i=1}^{k} w_i^2 \tilde{x}^t V_i \tilde{x} + 2\sum_{i,s=1}^{k} w_i w_s \tilde{x}^t V_{is} \tilde{x}},$$

$$\begin{aligned} subject\ to: \ & g_r(x) \le 0,\ r = 1,\ldots,m, \\ & x_j \in L_\alpha(\tilde{x}_j) \\ & x_j \ge 0,\ \ j = 1,\ldots,n \end{aligned} \tag{59}$$

The constraint $x_j \in L_\alpha(\tilde{x}_j)$ is equivalent to $x_{L_j} \le x_j \le x_{U_j}$, $j = 1,\ldots,n$ provided that x_{L_j} and x_{U_j} are lower and upper bounds respectively of the variables x_j .

So problem (59) can be written in the following form:

$$Minimize \quad \sum_{i=1}^{k} w_i e_i \tilde{x} + \Phi^{-1}(\beta)\sqrt{\sum_{i=1}^{k} w_i^2 \tilde{x}^t V_i \tilde{x} + 2\sum_{i,s=1}^{k} w_i w_s \tilde{x}^t V_{is} \tilde{x}},$$

$$\begin{aligned} subject\ to: \ & g_r(x) \le 0,\ r = 1,\ldots,m, \\ & x_{L_j} \le x_j \le x_{U_j},\ \ j = 1,\ldots,n \\ & x_j \ge 0. \end{aligned}$$

(60)

The optimal solution x^* of the deterministic programming problem(60) can be found easily by using any available programming package.

5.3 An illustrative Example:

This problem appears in Goicoechea et al. This problem is a multiobjective stochastic programming has three stochastic objectives and two crisp constraints. Its mathematical model as given in the source is:

$$Max \quad \hat{z}_1 = N(2,4)x_1 + N(3,2)x_2 \,,$$
$$Min \quad \hat{z}_2 = N(1,1)x_1^2 + N(1,1)x_2^2 \,,$$
$$Max \quad \hat{z}_3 = N(4,3)x_1 + N(1,1)x_2 \,,$$
$$subject\ to \quad x_1^2 + x_2^2 \le 25 \;;$$
$$3x_1 + x_2 \le 12 \;;$$
$$0 \le x_1 \le 4, \qquad 1 \le x_2 \le 5.$$

(61)

To obtain the deterministic equivalent of problem (61), katoaka's criterion and expected value standerd deviation efficiency applied , the first and the third objective is rewritten as

$$Min \quad \hat{z}_1 = -N(2,4)x_1 - N(3,2)x_2 \,,$$
$$Min \quad \hat{z}_3 = -N(4,3)x_1 - N(1,1)x_2$$

The deterministic equivalent of problem(1).

$$Min \left[w_1(-2x_1 - 3x_2) + w_2(x_1^2 + x_2^2) + w_3(-4x_1 - x_2) + \Phi^{-1}(\beta)\sqrt{w_1^2(4x_1^2 + 2x_2^2) + w_2^2(x_1^4 + x_2^4) + w_3^2(3x_1^2 + x_2^2)} \right]$$

$$subject\ to \quad x_1^2 + x_2^2 \le 25 \;;$$
$$3x_1 + x_2 \le 12 \;;$$
$$0.5 \le x_1 \le 4, \qquad 1 \le x_2 \le 5.$$

(62)

Let $\beta = 0.95,\ w = (0.2,\ 0.3,\ 0.5)^{T}$, $x^* = (1.21, 1)^{T}$

On the other hand, a fuzzy programming problem having fuzzy parameters in the objective function and the constraints can be formulated as follows:

$$Min \left[w_1(-2\tilde{x}_1 - 3\tilde{x}_2) + w_2(\tilde{x}_1^2 + \tilde{x}_2^2) + w_3(-4\tilde{x}_1 - \tilde{x}_2) + \Phi^{-1}(\beta)\sqrt{w_1^2(4\tilde{x}_1^2 + 2\tilde{x}_2^2) + w_2^2(\tilde{x}_1^4 + \tilde{x}_2^4) + w_3^2(3\tilde{x}_1^2 + \tilde{x}_2^2)} \right]$$

$$subject\ to \quad \tilde{x}_1^2 + \tilde{x}_2^2 \le 25 \;;$$
$$3\tilde{x}_1 + \tilde{x}_2 \le 12 \;;$$
$$0 \le \tilde{x}_1 \,,$$
$$0 \le \tilde{x}_2$$

(63)

The nonfuzzy $\alpha -$cut nonlinear programming problem equivalent to problem(63) can be written as:

$$Min \left[w_1(-2x_1 - 3x_2) + w_2(x_1^2 + x_2^2) + w_3(-4x_1 - x_2) + \Phi^{-1}(\beta)\sqrt{w_1^2(4x_1^2 + 2x_2^2) + w_2^2(x_1^4 + x_2^4) + w_3^2(3x_1^2 + x_2^2)} \right]$$

$$subject\ to \quad x_1^2 + x_2^2 \le 25 \;;$$
$$3x_1 + x_2 \le 12 \;;$$
$$0.5 \le x_1 \le 4, \qquad 1 \le x_2 \le 5.$$

The fuzzy parameters are characterized by the following fuzzy numbers

$\tilde{x}_1 = (0,1,3,5)$, $\tilde{x}_2 = (0,2,4,6)$

and we assume that the membership function corresponding to the fuzzy numbers x_j, $j = 1,2$ takes the form in equation:

$$\mu_{\tilde{x}_j}(x_j) = \begin{cases} 0 & x_j \le r_1, \\ 1 - \left(\dfrac{x_j - r_2}{r_1 - r_2}\right)^2 & r_1 \le x_j \le r_2, \\ 1 & r_2 \le x_j \le r_3, \\ 1 - \left(\dfrac{x_j - r_3}{r_4 - r_3}\right)^2 & r_3 \le x_j \le r_4, \\ 0 & x_j \ge r_4. \end{cases}$$

Conclusion:

In this paper, the possibilistic linear programming problem with exponential distribution function is converted to a usual mathematical programming problem based on maximizing the possibility measure, then the stochastic linear programming problem with multivariate normal distribution is treated using the probability maximization model. For such problems the stability set of the first kind is defined and characterized. The transformation between the possibilistic linear programming problem with exponential distribution function and the stochastic linear programming problem with multivariate normal distribution is discussed also.

Finally, it must be noted that, although the transformation method has been derived where both stochastic and possibilistic problems are linear, and with multivariate normal and exponential distributed variable respectively, the analysis for non linear with other type of variables is needed.

This, together with the application of this method to real problems will be analyzed in future works.

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