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A brief review of numerical methods for solving the boundary value problems of PDE

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Abstract. In science and engineering, partial differential equations (PDEs) are employed for modeling and comprehending an extensive variety of physical phenomena. Solving these equations analytically is complicated and requires a lot of research and time. Mesh-based and meshless techniques are two popular ways to solve PDEs numerically. Mesh-based methods depend on breaking up the computational domain into a structured or unstructured mesh. These methods are accurate and based on well-established theories. However, they often have challenges with complex geometries, flexibility, and the high cost of computation that comes with mesh generation and refinement. On the other hand, meshless methods are a different way to do things that don't require meshing. Instead, these methods use a number of points that are spread out to get close to the solution. It can handle complex geometries, is easy to implement, and is easier to deal with problems that have boundaries or interfaces that change. This paper provides a summary of solving PDEs using both mesh-based and meshless approaches, with a focus on elasticity implementation. In addition to explaining the characteristics of each of the two numerical methods.

1 Introduction

Partial differential equations (PDEs) are exploited a lot in science, engineering, and mathematical fields because they are a powerful way to describe and analyze a wide range of phenomena and processes. Solving the PDEs is done by two ways:analytically via separation of variables [1], transformation [2, 3, 4], power series [5], or using Green's theory [6]. But there are generally limited and simple ways to solve 1D or 2D problems. In real life, the problems are complex and irregular, making them challenging to derive and solve analytically. As a result, numerical solutions were developed. The numerical solutions investigate complex phenomena, accurate modeling of real-world systems, and the exploration of scenarios that are not possible to address analytically. The numerical solutions are divided into two methods, mesh-based and meshless methods.

Mesh-based methods are the most common approach to solve PDEs. In these methods, the domain of the PDE is discretized using a mesh, which is a collection of lines, triangles, or any shapes. The most common mesh-based methods are the finite difference, finite element and boundary element method.

The finite difference method (FDM) [7] is one of the mesh methods to solve boundary value problems (BVPs) in the literature. In this method the PDEs are approximated to difference equations. Hao et al [8] presented an algorithm to solve BVPs with non-smooth solutions based on finite difference schemes, which improved accuracy for second-order. Hao et al [9] extended the modified algorithm to deal with higher orders.

The finite element method (FEM) [10] is extensively used in engineering and applied mathematics to solve PDEs. In this method, the domain is discretized into elements, and using the weighted residual technique, the governing equations are converted to integral equations. It offers several advantages, including flexibility in handling complex geometries, the ability to incorporate different types of boundary conditions, and scalability for large-scale problems. Gong et al [11] investigated the optimal convergence properties of FEMs for an elliptic distributed optimal control problem, providing a priori and a posteriori error estimates with explicit dependence on the regularization parameter and proving rate optimality for an adaptive algorithm. Extensive numerical experiments validate the theoretical results presented. Wang et al [12] modified FEM for solving elasticity problems in a mixed form. This technique aimed to decrease degrees of freedom by representing boundary

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stress tensors as averages of interior stress tensors, while also providing optimal error estimates. Supporting this technique through numerical experiments demonstrates the method's efficiency and accuracy.

The boundary element method (BEM) [13] is a semi-analytical approach that transforms equilibrium equations into integral equations, with the integrals evaluated along the problem boundary. In order to compute these integrals, the boundary is discretized. Li et al [14] developed a quadrature method based on BEM for solving 2D elasticity problems and compared the results with other numerical methods.

Meshless methods are approaches to solve PDEs where there is no need to discretize the domain of the PDE. Instead, the PDE is approximated directly at a set of points, which are called collocation points. There are many different types of meshless methods, including radial basis function (RBF) [15] methods, moving least squares (MLS) [16] methods, Method of fundamental solutions (MFS), and natural neighbour (NN) [17] methods. Each type has its own advantages and disadvantages. The MFS research topic continues to gain attention, as evidenced by recent review papers [18, 19]. Markous et al [20] developed a meshless method to solve 2D elasticity problems by using distributed source located at the domain boundary. Aly et al [21] solved the couple stress elasticity by using the meshless technique and added the effect of body force as a particular solution. Chen et al [22] used the effective condition number to determine the effectiveness of the source point location on which the solution depends.

In recent years, there has been a coupling between mesh and meshless methods for solving PDEs [23, 24] combining the strengths of both approaches to overcome their limitations. Mesh-based methods provide structured discretization and efficient handling of complex geometries, whereas meshless methods are more flexible in handling irregular geometries and adaptable to changing domains. Figure. 1 shows the different types of solution of PDE analytical and numerical.

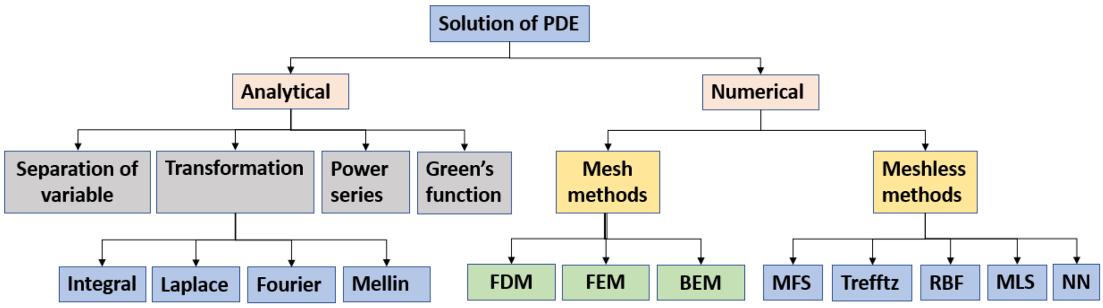

Figure. 1 Types of solution for partial differential equation.

2 Basic equations and definitions

The general second order PDEs can be written in the following form [25]:

$$A\frac{\partial^2}{\partial x^2}u(x,y)+B\frac{\partial^2}{\partial x\partial y}u(x,y)+C\frac{\partial^2}{\partial y^2}u(x,y)=f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) \tag{1}$$

where A, B, and C are arbitrary functions.
Eqn. (4) is classified into three categories depending on the sign of Δ as shown:

$$\Delta=\begin{cases} +ve & \textbf{Hyperbolic}\text{ equation.}\\ 0 & \textbf{Parabolic}\text{ equation.}\\ -ve & \textbf{Elliptic}\text{ equation.}\end{cases} \tag{2}$$

where,

$$\Delta=B^2-4AC \tag{3}$$

Hyperbolic PDEs include wave like behavior and solutions that propagate along characteristic curves. Some examples where the hyperbolic PDEs appears are the propagation of sound waves, seismic waves, electromagnetic waves, also, in fluid dynamic problems describing compressible and incompressible flow steady and unsteady and turbulent flows. In Electromagnetic waves Maxwell's equations which govern the behavior of electromagnetic fields, is a hyperbolic PDEs, these equations are essential in electromagnetics, optics, and the design of communication systems. Seismology and Earthquake Modeling are used extensively to model the propagation of seismic waves generated by earthquakes. Aerospace Engineering employed to model the noise generated by aircraft and rockets, is another application to hyperbolic PDEs helping engineers to design quieter and more efficient aerospace systems. Plasma physics study the behavior of charged particles and electromagnetic fields in high-temperature plasmas used in fusion research and space physics.

Parabolic PDEs involve gradual change and time-dependent evolution. Examples of parabolic PDEs are: Heat equation which describes the transient behavior of temperature distributions in materials over time due

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to heat conduction. Applications include thermal analysis in materials science. Diffusion and Mass Transport used to model the diffusion of substances, such as chemical solutes or particles, in various media. Modeling pollutant dispersion in the environment and drug diffusion in biological tissues.

Elliptic PDEs are characterized by their smooth and continuous solutions, and their applications often involve equilibrium or steady-state situations. These equations are employed in structural mechanics to analyze the equilibrium state of structures under loads. Examples include finding the deformation and stress distribution in structures subjected to external forces, and also used in image processing for tasks like image smoothing and denoising. The solutions to these equations can help preserve important features while reducing noise.

In mathematical modeling, the distinction between initial-value problems (IVPs) and BVPs is critical for understanding and solving differential equations. An initial-value problem specifies a system's behavior at a single point or over a finite interval at the beginning of time. This typically entails specifying the system's initial state, such as its position, velocity, or temperature, which serves as the starting point for the evolution of the system's behavior over time. In contrast, a BVPs involves specifying a system's behavior along the boundaries of a domain or region rather than at a single point in time.

3 Basic equations of elasticity problem

Elasticity problems form the basis of material mechanics, structural mechanics, and other branches of engineering. Assume a solid exists in a region $\Omega \in R^2$ where the linear elasticity problem is being considered. The degrees of freedom for the plane strain are two in-plane displacements in x and y- direction.

In mechanical theory of elasticity, the governing PDE equation is ([26]):

$$\sigma_{\beta\alpha,\beta} = 0 \tag{4}$$

where, $\sigma_{\alpha\beta}$ is the stress tensor, and $(.)_{,\alpha} = \frac{\partial(.)}{\partial x_\alpha}$. The Greek indices are assumed to take values (1,2). The stress tensor for generalized force is provided [20] as follows:

$$\sigma_{\beta\alpha} = \lambda e_{\gamma\gamma}\delta_{\alpha\beta} + 2Ge_{\alpha\beta} \tag{5}$$

where, $\delta_{\alpha\beta}$, $e_{\alpha\beta}$, λ, and G are Kroneker's delta, the strain tensor, Lamé's first parameter, and shear modulus respectively. The fundamental solution of elasticity for displacement is as follows:

$$U_{\alpha\beta} = \frac{1}{8\pi \mathrm{G}(1-v)}\left\{-\delta_{\alpha\beta}(3-4v)\ lnr + r_{,\alpha}r_{,\beta}\right\} \tag{6}$$

and the fundamental solution of elasticity for traction is as follows:

$$T_{\alpha\beta} = \frac{1}{8\pi(1-\nu)r}\left[(2-4\nu)r_{,\alpha}n_\beta + (-2+4\nu)\delta_{\alpha\beta}r_{,n}(-2+4\nu)r_{,\beta}n_\alpha - 4r_{,\alpha}r_{,\beta}r_{,n}\right] \tag{7}$$

4 Finite difference method

The finite difference method (FDM) is easy to implement, both mathematically and in terms of coding, especially in academia, where simple model problems are frequently used for educational and research purposes. This method depends on dividing the spatial domain of the PDE into a uniformly partitioned mesh as shown in Fig. 2. In this method the PDE is approximated to difference equations using Taylor series. The choice of finite difference scheme (e.g., forward, backward, or central differences) depends on the specific PDE and problem. The boundary conditions are applied the grid points.

From Taylor series, the displacement function $u(x, y)$ is as follows:

$$\nabla^2 u_{ij} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \tag{8}$$

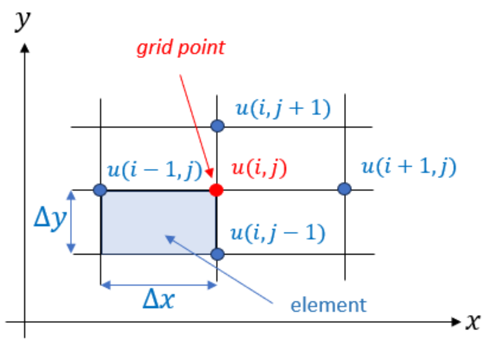

Figure. 2 Finite difference mesh for 2D.

5 Finite element method

The Finite Element Method (FEM) is a powerful and versatile numerical tool that has become standard in engineering analysis and design due to its ability to handle complex geometries and diverse physical phenomena and also its variety in the form of the element shapes as shown in Fig. 3. It is widely used in engineering for solving problems in structural analysis, heat transfer, fluid dynamics, and electromagnetics, among other fields. In this method the domain of the problem is discretized into elements and then the governing PDE is converted to integral equations using the weighted residual technique. In structural analysis, the most common problem is to determine the displacement or deflection acting from a set of static or dynamic loads. The final equation is in the following form

$$\{F\} = [K]\{u\} \tag{9}$$

where, $\{F\}$ is the vector of loads, $\{u\}$ is the vector of displacements, and $[K]$ is the stiffness matrix.

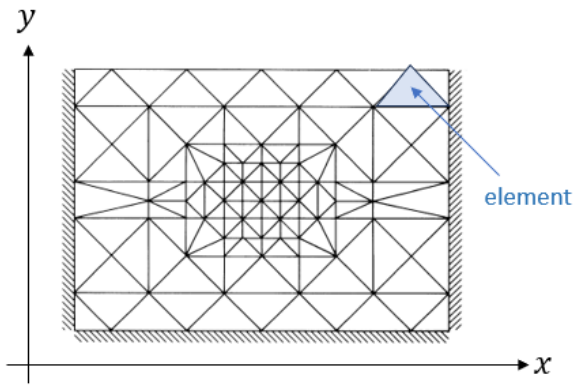

Figure. 3 Finite element mesh for 2D.

6 Boundary element method

The Boundary Element Method (BEM) is a semi-analytical method for solving PDEs by transforming them into integral equations over the boundary of the domain. One of the main advantages of BEM is that it doesn't require the discretization of the entire domain, only the boundaries. The element in this method has a one-degree less dimension than the problem. This could result in substantial computational efficiencies when compared to alternative numerical techniques, especially for problems with complex geometries. As an example, in elasticity the governing PDE is converted to the following integral equation:

$$c_{\alpha\beta}(\xi)u_\beta(\xi) = \int_\Gamma U^*_{\alpha\beta}(\xi, \mathrm{x})t_\beta(\mathrm{x})d\Gamma(\mathrm{x}) - \int_\Gamma T^*_{\alpha\beta}(\xi, \mathrm{x})u_\beta(\mathrm{x})d\Gamma(\mathrm{x}) \tag{10}$$

where $u_\beta(\mathrm{x})$, $t_\beta(\mathrm{x})$ are the displacement and traction at any field point x in β direction, and

$$c_{\alpha\beta}(\xi) = \begin{cases} 0 & \xi \text{ outside } \Omega \\ 1 & \xi \text{ inside } \Omega \\ \text{Get it using rigid body consideration} & \xi \text{ on } \Gamma \end{cases} \tag{11}$$

7 Meshless methods

Meshless methods offer an appealing approach for solving PDEs without requiring discretization, relying solely on boundary points placement. Among these methods, MFS stands out, tackling a virtual problem embedded within an infinite domain rather than directly addressing the real problem, as shown in Fig. 4. The displacement and traction at any point are known as:

$$u_\alpha(\mathbf{x}) = \sum_{k=1}^{k=M} U^*_{\alpha\beta}(\mathbf{x}, \xi_k) \cdot \psi_\beta(\xi_k) \tag{12}$$

$$t_\alpha(\mathbf{x}) = \sum_{k=1}^{k=M} T^*_{\alpha\beta}(\mathbf{x}, \xi_k) \cdot \psi_\beta(\xi_k) \tag{13}$$

where $U^*_{\alpha\beta}(\mathbf{x}, \xi_k)$, and $T^*_{\alpha\beta}(\mathbf{x}, \xi_k)$ are the fundamental solutions, and M is the number of source points.

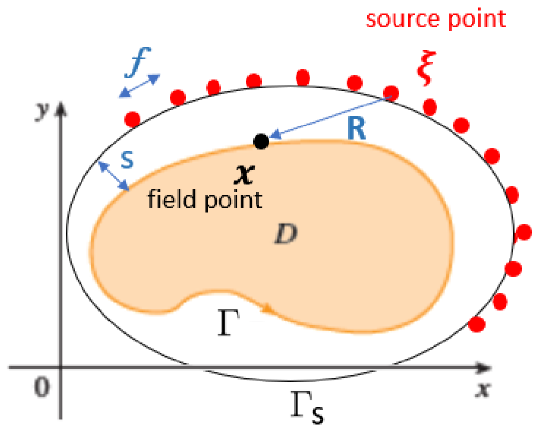

Figure. 4 Boundary element method and method of the fundamental solution in 2D.

8 Numerical example

In this section the four above techniques are applied to a numerical example (2D elasticity) and the results are compared. In this numerical example, a square plate with a side length of $L = 6mm$ is subjected to a constant traction of $P = 1MPa$ along its right side as shown in Fig. 5. The material properties are the Poisson's ratio $\nu = 0.3$ with different values of shear modulus G are considered. The problem is solved under plane strain conditions.

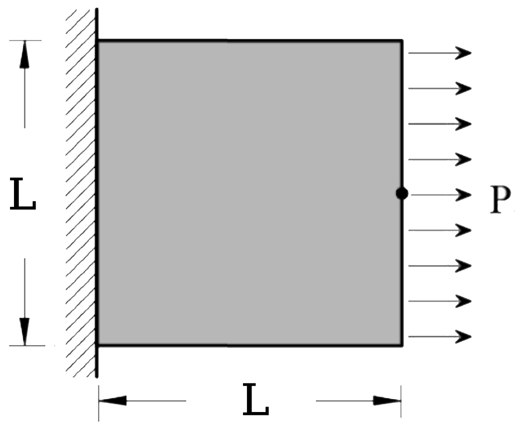

Figure. 5 Square plate under tension.

8.1 Method 1: finite difference method

This example is solved by FDM using a total number of square elements $N = 100$ [0.6×0.6mm]

8.2 Method 2: finite element method

This example is solved by FEM using the discretization in FDM in order to compare the results.

8.3 Method 3: boundary element method

This example is solved by BEM using ten elements (quadratic element) per each edge with length of 0.6 mm

8.4 Method 4: method of fundamental solution

This example is solved by MFS using a total number of source points of $M = 40$ where sources are located at distance S (length between Γ_s and Γ)= $4f$ (length between two adjacent source points)

Table 1 The values of displacement $u_1(mm)$, and $u_2(mm)$.

G (MPa)	FDM		FEM		BEM		MFS	
	u_1	u_2	u_1	u_2	u_1	u_2	u_1	u_2
0.5	4.7982	1.6978	4.7914	1.7188	4.8703	1.7278	4.8803	1.7329
1	2.4411	0.8542	2.4324	0.7988	2.4502	0.8653	2.4401	0.8664
1.5	1.6241	0.5741	1.6235	0.5759	1.6254	0.5762	1.6268	0.5776
2	1.2219	0.4319	1.2192	0.4319	1.2201	0.4324	1.2201	0.4332
2.5	0.9757	0.3462	0.9759	0.3461	0.9761	0.3463	0.9761	0.3466

The results shown in Table.1 demonstrates the values of displacement at the x-direction u_1 at the midpoint of the right side, and the values of displacement at the y-direction at the midpoint of the lower side by different numerical methods.

9 Conclusions

Due to the difficulty of solving boundary value problems with analytical methods, researchers have resorted to numerical methods. The finite difference method divides the problem into equal elements, and all grid points have the same spacing in each direction, so it is preferred in academic studies and problems with regular domains. As for the finite element method, the division of the issue area according to importance is therefore used more in life applications, and there are many packages that exploit this method, but it takes more computation time. The boundary element method is characterized by reducing the degree of the problem in discretization, which facilitates the solution process and requires less computation time than in finite element, but after a difficult mathematical derivation. The meshless method is characterized by not discretizing the problem; in addition, it avoids singularity and has a good consequence for the effectiveness of the source location. A numerical example is solved to compare the results of different numerical techniques.

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