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The solution of Rough Bilevel Nonlinear Programming Problem by using Trust-Region Penalty Method

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Abstract. This paper, the rough bilevel nonlinear programming problem (RBNPP) is discussed taking into consideration which level is more important than the other. BNPP is transformed into a crisp unconstrained programming problem. A trust-region method is used to ensure the global convergence of the algorithm. The mechanism of solving RBNPP is presented. There are many situations of roughness in these problems are discussed. The solution procedures for solving all roughness situations are introduced based on the new proposed methodology. The definitions of solutions are defined in all different situations. Also, we show the definition of the fully optimal solution of the BNPPs. Finally, numerical examples are given to show solution procedures of a RBNPP based on the new methodology.

1. Introduction

Bilevel problems (BLP) are one of the most interesting branches of mathematical programming programs see e.g., [1, 2].

Karush-Kuhn-Tucker (KKT) approach [3] is an interesting method to deal with BLP. The lower level is replaced by its the KKT-conditions. The objective function of the 2nd level sometimes is very important and must be satisfied first. So, a new methodology treats BLP depend on which level is important to decision maker (DM) . If the 1st level superior, then the 2nd level programming problem is replaced with its KKT conditions. Otherwise, if the 2nd level is superior, the 1st level is replaced with its KKT conditions. So, the suggestion here is solving BLP twice, one as 1st level is superior and the 2nd level is superior. If the solution of both is the same, the full optimal solution of BLP is gotten. This suggestion is very important in real applications.

Moreover, from a topological viewpoint, BLP are more complicated than standard finite programming. The feasible set of a BLP may for example not be closed [4]. So, this paper used a numerical approach combines the penalty method with trust region method to solve the modified BLP. This problem converted to an unconstrained programming problem. The decisions in real-world situations are always made based on imprecision (incomplete) information. There are many approaches are used to deal with incomplete information like fuzzy [5], rough [6,7] ...etc. These approaches based on the kind of approximation which is

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convenient and enough for making good decisions. Rough set theory (RST) is one of the most interesting approaches to deal with vagueness and imprecision [8].

The rough programming problems are presented in [9-13]. It is divided into three types based on where roughness is found. The aim of the present article is presenting a procedure for treating BLP with different roughness situations. The definitions of solutions are defined in all situations. The effectiveness of our technique is shown and presented by different examples.

2.Bilevel problem Formulation

BLP have the following form:

$$min_x \, f_U(x,y)\,, \tag{1}$$

S.T:

$$g_{U_i}(x,y) \le 0\,, i = 1,2,\dots,m_1$$

$$\min_y f_L(x,y)$$

S.T:

$$g_{L_j}(x,y) \le 0, j = 1,2,\dots,m_2$$

where, $f_U(x,y)$,$f_L(x,y)$, $g_U(x,y)$, and , $g_L(x,y)$ are continuous and differentiable functions.

Different approaches are suggested to solve BLP, [11, 14, 15]. The conventional solution approach transformed BLP to programming problem. Assume that the 1st level superior , the 2nd level programming problem is replaced with its KKT conditions [16, 17]. This approach is used to convert BLP to NPP which solved by using a numerical method. The KKT conditions of the 2nd level for BLP are:

$$\begin{gathered}\nabla_y f_{\mathrm{L}}(x,y) + \mu \nabla_y \overline{g}_L(x,y) = 0,\\ \overline{g}_L(x,y) \le 0,\\ \mu_j \overline{g}_{L_j}(x,y) = 0, j = 1,\dots,m_2,\\ \mu_j \ge 0, j = 1,\dots,m_2,\end{gathered} \tag{2}$$

where $\mu \in \Re^{m_2}$ is a Lagrange multiplier vector, see [15]. Then, Problem (1) is :

$$\begin{array}{ll}\min\limits_{x,y} & f_U(x,y)\\ s.t. & g_U(x,y) \le 0,\\ & \nabla_y f_L(x,y) + \nabla_y g_L(x,y)\mu = 0,\\ & \overline{g}_L(x,y) \le 0,\\ & \mu_j \overline{g}_{L_j}(x,y) = 0, j = 1,\dots,m_2\\ & \mu_j \ge 0, j = 1,\dots,m_2.\end{array} \tag{3}$$

Now, in case of the 2nd level is suporior, problem (1) is:

$$\begin{array}{ll}\min\limits_{x,y} & f_L(x,y)\\ s.t. & g_L(x,y) \le 0,\\ & \nabla_x f_U(x,y) + \nabla_x g_U(x,y)\mu = 0,\\ & \overline{g}_U(x,y) \le 0,\\ & \mu_j \overline{g}_{U_i}(x,y) = 0, i = 1,\dots,m_1\\ & \mu_j \ge 0, j = 1,\dots,m_1.\end{array} \tag{4}$$

Remark:

It is preferred to solve both problems (4) & (5) and compare the solutions of both problems. If the solutions of problems (4) & (5) are the same, then the full optimal solution is found. The best one or give all solutions to DM that choose the suitable solution. This idea is shown in examples 1 &2.

Definition 1:
The optimal solution of problem (1) is said to be full optimal solution if the optimal solution of problem (4) & problem (5) is the same.

Let us describe the methodelgy that combine the penenlty method with trust region method on solving one of the reduced poblems . The problem (4) can be summarized as:

$$\begin{aligned} \min_{\bar{x}} \quad & f_U(\bar{x}) \\ s.t. \quad & D_e(\bar{x}) = 0, e \in E, \\ & D_i(\bar{x}) \leq 0, i \in I, \end{aligned} \tag{5}$$

where $\bar{x} = (x, y, \mu)^T$,$E = \{1, \ldots, \mathrm{L} + m_2\}$, $I = \{1, \ldots, \mathrm{U} + 2\mathrm{L}\}$, and $E \cap I = \emptyset$. Suppose that $f_U(\bar{x})$, $D_e(\bar{x})$ for all $e \in E$, and $D_i(\bar{x})$ for all $i \in I$ are at least twice continuously differentiable functions.

Motivated by the active-set mechanism in [2], a 0-1 diagonal matrix $Z(\bar{x})$ is defined, whose diagonal entries are

$$z_i(\bar{x}) = \begin{cases} 1 & if \ \ i \in \ \ E, \\ 1 & if \ \ D_i(\bar{x}) \geq \ \ 0 \ \ for \ aLL \ \ i \in \ \ I, \\ 0 & if \ \ D_i(\bar{x}) \ \ < \ \ 0 \ \ \ for \ \ aLL \ \ i \in \ \ I. \end{cases} \tag{6}$$

The equality constrained optimization (ECO) problem is Prpblem (6) by using $Z(\bar{x})$

$$\begin{aligned} \min \quad & f_U(\bar{x}) \\ s.t. \quad & P(\bar{x})^T Z(\bar{x}) P(\bar{x}) = 0, \end{aligned}$$

where $P(\bar{x})$ is the vector function; for more details see[22]. A penalty method ([13], [14]), is used to make the above problem into unconstrained problem.

$$\begin{aligned} \min \quad & f_U(\bar{x}) + \frac{\rho}{2} \parallel Z(\bar{x}) P(\bar{x}) \parallel^2 \\ s.t. \quad & \bar{x} \in \Re^{m_1 + m_{2+\mathrm{L}}}, \end{aligned} \tag{7}$$

where $\rho \in \Re$ is a positive parameter.

The first-order necessary condition for the point $\bar{x}_*$ to be a local minimizer of Problem (8) is

$$\nabla_{\bar{x}} f_U(\bar{x}_*) + \rho \nabla P(\bar{x}_*) Z(\bar{x}_*) P(\bar{x}_*) = 0. \tag{8}$$

If the point $\bar{x}_*$ meets the first-order necessary conditions of Problem (6) [20], then it satisfies the condition (9) of Problem (8) but the converse is not necessarily true. The proposed algorithm shows that if $\bar{x}_*$ interpolates the first-order necessary condition of Problem (8), then it also achieves the first-order necessary conditions of Problem (6). The suggested trust-region for solving to combine the sequential quadratic programming (SQP) method with the trust-region idea [21]. This method for Problem (8) is

$$\begin{aligned} \min \quad & q_k(s_k) = f_{U_k} + \nabla f_{U_k}^T s + \frac{1}{2} s^T H_k s + \frac{\rho_k}{2} \parallel Z_k (P_k + \nabla P_k^T s) \parallel^2 \\ s.t. \quad & \parallel s \parallel \leq \delta_k, \end{aligned} \tag{9}$$

where the radius of trust-region is $\delta_k > 0$, and the Hessian of $f_U(\bar{x}_k)$, $H_k$or an approximation to it. $f_{U_k} = f_U(\bar{x}_k)$, $P_k = P(\bar{x}_k)$, $\nabla f_{U_k} = \nabla f_U(\bar{x}_k)$, $\nabla P_k = \nabla P(\bar{x}_k)$, $Z_k = Z(\bar{x}_k)$ and so on. The trust-region algorithm solved problem (4) as follows.

The trial step s_k is esitmated by using A conjugate gradient method [22] . It is perffered when the Hessian is indefinite or in large-scale problems.

The subproblem (10) is solved by :

Algorithm 1 *: (Evaluate s_k)*

Step 1. Set $0 = s_0 \in \Re^{n_1+n_2+m_2}$, $w_0 = -(\nabla f_{U_k} + \rho_k \nabla P_k Z_k P_k)$, and $v_0 = w_0$.

Step 2. For $j = 1, \dots, (n_1 + n_2 + m_4)$ do

Compute $B_k = H_k + \rho_k \nabla P_k Z_k \nabla P_k^T$.

Compute $c_j = \frac{w_j^T w_j}{v_j^T B_k v_j}$.

Compute γ_j such that $\| s_j + \gamma_j v_j \| = \delta_k$.

If $v_j^T B_k v_j \leq 0$, then set $s_k = s_j + \gamma_j v_j$ and Stop.

Else, set $s_{j+1} = s_j + c_j v_j$ and

$w_{j+1} = w_j - c_j B_k v_j$.

If $\frac{w_{j+1}}{w_0} \leq \varepsilon_0$, set $s_k = s_{j+1}$ and Stop.

Compute $\bar{q}_j = \frac{w_{j+1}^T w_{j+1}}{w_j^T w_j}$ and the new direction is

$v_{j+1} = w_{j+1} + \bar{q}_j v_j$.

The following merit function is tested s_k is accepted or not

$$\ell(\bar{x}_k; \rho_k) = f_U(\bar{x}_k) + \frac{\rho_k}{2} \| Z(\bar{x}_k) P(\bar{x}_k) \|^2. \tag{10}$$

An actual reduction $Ared_k$ and a predicted reduction $Pred_k$ in the merit function is used to test $\bar{x}_{k+1} = \bar{x}_k + s_k$ takes as a next iterate or not. $Ared_k$ (11) is evaluted as follows

$$Ared_k = f_U(\bar{x}_k) - f_U(\bar{x}_{k+1}) + \frac{\rho_k}{2} [\| Z_k P_k \|^2 - \| Z_{k+1} P_{k+1} \|^2], \tag{11}$$

and $Pred_k$ is defined as

$$Pred_k = q_k(0) - q_k(s_k) \tag{12}$$

$$= -\nabla f_{U_k}^T s_k - \frac{1}{2} s_k^T H_k s_k + \frac{\rho_k}{2} [\| Z_k P_k \|^2 - \| Z_k (P_k + \nabla P_k^T s_k) \|^2]. \tag{13}$$

Algorithm 2 *: (Test s_k and update , δ_k algorithm)*

Choose $0 < \tau_1 < \tau_2 \leq 1$, $\delta_{\max} > \delta_{\min}$, and $0 < \eta_1 < 1 < \eta_2$.

Let $r_k = \frac{Ared_k}{Pred_k}$.

While $r_k < \tau_1$, or $Pred_k \leq 0$.

Set $\delta_k = \eta_1 \| s_k \|$.

Evaluate a new s_k.

If $\tau_1 \leq r_k < \tau_2$, then set $\bar{x}_{k+1} = \bar{x}_k + s_k$.

$\delta_{k+1} = \max(\delta_k, \delta_{\min})$.

End if.

If $r_k \geq \tau_2$, then set $\bar{x}_{k+1} = \bar{x}_k + s_k$.

$\delta_{k+1} = \min\{\delta_{\max}, \max\{\delta_{\min}, \eta_2 \delta_k\}\}$.

End if.

A scheme suggested by [22] updated the parameter $\rho_k > 0$ which presented in the following algorithm.

Algorithm 3 :

Compute $Pred_k$ given by (15).

If $Pred_k \geq \| \nabla P_k Z_k P_k \| \min\{\| \nabla P_k Z_k P_k \|, \delta_k\}$. (14)

Set $\rho_{k+1} = \rho_k$.

Else, set $\rho_{k+1} = 2\rho_k$.

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End if
The stopping criteria when either $\| \nabla f_{U_k} \| + \| \nabla P_k Z_k P_k \| \leq \varepsilon_1$ or $\| s_k \| \leq \varepsilon_2$ for some tolerances $0 < \varepsilon_1$ and $0 < \varepsilon_2$.

The steps of suggested mehod:

Algorithm 4

Step 0: Given $\bar{x}_0 \in \Re^{(n_1+n_2+m_2)}$. Choose $0 < \varepsilon_1$, $0 < \varepsilon_2$, τ_1, τ_2, η_1, and η_2, such that $0 < \tau_1 < \tau_2 \leq 1$ and $0 < \eta_1 < 1 < \eta_2$. Choose $\delta_{\min}$, $\delta_{\max}$, and δ_0 such that $\delta_{\min} \leq \delta_0 \leq \delta_{\max}$. Set $\rho_0 = 1$. Set $k = 0$.
Step 1: If $\| \nabla f_{U_k} \| + \| \nabla P_k Z_k P_k \| \leq \varepsilon_1$, then stop the algorithm.
Step 2: Using Algorithm (1) to compute s_k.
Step 3: If $\| s_k \| \leq \varepsilon_2$, then the algorithm stops.
Step 4: Set $\bar{x}_{k+1} = \bar{x}_k + s_k$.
Step 5: Compute Z_{k+1} given by (8).
Step 6: Test the step and update using Algorithm (2).
Step 7: Update ρ_k using Algorithm (3).
Step 8: Set $k = k + 1$ and go to Step 1.

3. Numerical examples

The idea of the new methodology can be presented in the following examples:

Example 1:

$$\min_{x_1} f_U = x_1^3 x_2 + x_3$$
$$\text{S.T:} \quad x_1 + 2x_2 + x_3 \leq 6,$$
$$x_1^2 + x_2 + (x_3 - 1)^2 \leq 6,$$
$$x_1 \geq 0$$
$$\min_{x_2, x_3} x_1 + x_2^2 + x_3 + 2$$
$$\text{S.T} \quad x_1 + x_2 + 3x_3 \leq 10,$$
$$-x_1 + x_2^3 + 2x_3 \leq 2,$$
$$x_2 \geq 0, x_3 \geq 0$$

By using the reduction formulas, and solving them using algorithm 4 the optimal solution of problems (4)&(5) is

$x_1 = 0, x_2 = 0, x_3 = 0.$

Example 2:

$$\min_{x_1, x_2} f_U = (x_1 - 3)^2 + x_2 + (x_3 - 2)^2$$
$$\text{S.T:} \quad x_1 + x_2^2 - 2x_3 + 6 \geq 0,$$
$$x_1 + x_2 + x_3 - 9 \leq 0,$$
$$x_1 \geq 0, x_2 \geq 0$$
$$\min_{x_3} f_L = x_1 - 3x_2 + (x_3 - 5)^2$$
$$\text{S.T:} \quad x_1 + 2x_2 + x_3^3 \leq 12,$$
$$x_3 \geq 0$$

By using the reduction formulas the problem is transformed to a single programming problems, and solving them using algorithm 4 the optimal solution of problems (4) is (3.25,4.368,0.224), $f_U = 7.514, f_L = 12.766$. And The solution of problem by using problem formulation (5) is (3,0,3),$f_U = 0, f_L = 7$. As shown the solution from formulation problem (4) is better than the solution found from problem(5).

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4. Rough bilevel nonlinear programming problem:

RBNPP consist of two levels, namely, the 1st and/or 2nd levels each having its rough function and/or constraints are rough set is presented as:

$$\min_{x} f_U(x,y)$$
$$\text{S.T:} \quad x \in g_U(x,y) \tag{15}$$
$$\min_{y} f_L(x,y)$$
$$\text{S.T:} \quad y \in g_L(x,y)$$

where, $\underline{f}_U(x,y) \le f_U(x,y) \le \overline{f}_U(x,y)$,$\underline{f}_L(x,y) \le f_L(x,y) \le \overline{f}_L(x,y)$,$\underline{g}_U(x,y) \subseteq g_U(x,y) \subseteq \overline{g}_U(x,y)$,$\underline{g}_L(x,y) \subseteq g_L(x,y) \subseteq \overline{g}_L(x,y)$, and $\underline{f}_U(x,y), \overline{f}_U(x,y), \underline{f}_L(x,y), \overline{f}_L(x,y)$, $\underline{g}_U(x,y)$, $\overline{g}_U(x,y), \underline{g}_L(x,y), ,and\ \overline{g}_L(x,y)$ are functions at least first continuously differentiable functions.

The solution procedure and solution definitions of different situations of roughness in BLP are presented.

4.1 RBNPP when roughness on the constrains only:

$RBNPP_c$ is $RBNPP$ are deterministic is defined:

$$\min_{x} f_U(x,y)$$
$$\text{S.T:} \quad x \in g_U(x,y) \tag{16}$$
$$\min_{y} f_L(x,y)$$
$$\text{S.T:} \quad y \in g_L(x,y)$$

where, $\underline{g}_U(x,y) \subseteq g_U(x,y) \subseteq \overline{g}_U(x,y)$,$\underline{g}_L(x,y) \subseteq g_L(x,y) \subseteq \overline{g}_L(x,y)$, and $f_U(x,y), f_L(x,y)$, $\underline{g}_U(x,y), \overline{g}_U(x,y), \underline{g}_L(x,y)$, and $\overline{g}_L(x,y)$ are functions assumed to be at least first continuously differentiable functions. The classical idea divided the problem into four subproblems and solved them individually as shown in table 1.

Table 1. RBNPP when roughness on the constrains and objective function is deterministic

CP_1	CP_2	CP_3	CP_4
$\min_x f_U(x,y)$ S. T: $\overline{g}_U(x,y) \le 0$	$\min_x f_U(x,y)$ S. T: $\underline{g}_U(x,y) \le 0$	$\min_x f_U(x,y)$ S. T:$\overline{g}_U(x,y) \le 0$	$\min_x f_U(x,y)$ S. T:$\underline{g}_U(x,y) \le 0$
$\min_y f_L(x,y)$ S. T:$\overline{g}_L(x,y) \le 0$	$\min_y f_L(x,y)$ S. T:$\overline{g}_L(x,y) \le 0$	$\min_y f_L(x,y)$ S. T:$\underline{g}_L(x,y) \le 0$	$\min_y f_L(x,y)$ S. T:$\underline{g}_L(x,y) \le 0$

Let us show the procedure of solving $RBNPP_c$ at flowchart 1.

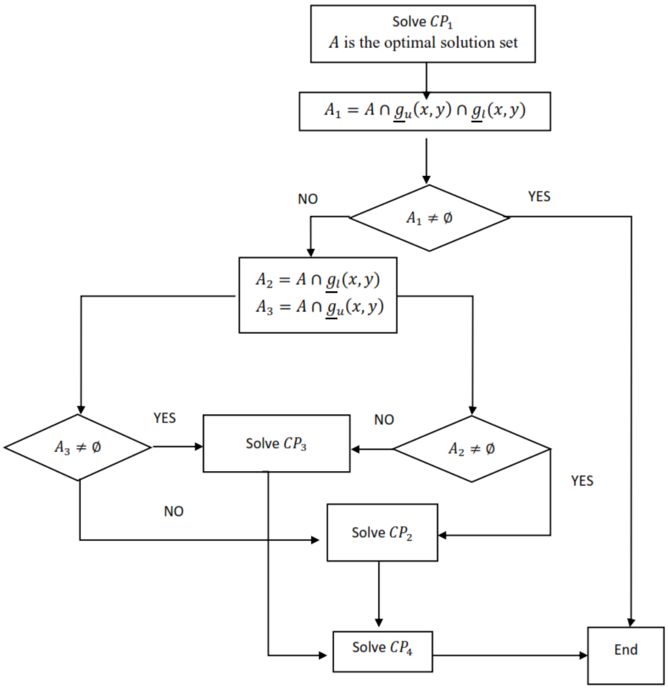

Figure 1. Flowchart 1 (Procedure solution for $RBNPP_c$)

Definition 2: *The surely optimal solution set for $RBNPP_c$*. The optimal solution set A_1 $\left(A_1 = A \cap \underline{g_U}(x,y) \cap \underline{g_L}(x,y)\right)$ is the surely optimal solution set of the problem (16), if $A_1 \neq \emptyset$.

Definition 3: *The surely optimal solution set to 1st level and possibly optimal solution to 2nd level for $RBNPP_c$* . A_3 is the surely optimal solution set to 1st level and possibly optimal solution to 2nd level where A_3 is the optimal solution set of CP_3 if $A_3 \neq \emptyset$.

Definition 4: *The possibly optimal solution set to 1st level and surely optimal solution to 2nd level for $RBNPP_c$*. The optimal solution set of CP_2 is A_2 is called the possibly optimal solution set to 1st level and surely optimal solution to 2nd level if $A_2 \neq \emptyset$.

Definition 5: *The possibly optimal solution set for $RBNPP_c$*. The optimal solution set A_4 is the possibly optimal solution set to 1st level and possibly optimal solution to 2nd level if $A_4 = A \cap \overline{g}_U(x,y) \cap \overline{g}_L(x,y), A_4 \notin \underline{g_U}(x,y) \cap \underline{g_L}(x,y)$.

Example 3:

$$\min_{x_1,x_2} f_U = 16x_1^2 + 9x_2^2$$

S.T:

$$(x_1,x_2) \in g_U(x_1,x_2)$$
$$\min_{x_3} f_L = (x_1 + x_2 - 5)^2$$

S.T

$$(x_1,x_2) \in g_L(x_1,x_2)$$

where $\underline{g_U}(x_1,x_2) \subseteq g_U(x_1,x_2) \subseteq \overline{g}_U(x_1,x_2)$,

$\overline{g}_U(x_1,x_2) = \{(x_1,x_2) \in R^2 | -5x_1 + x_2 - 1 \leq 0, x_1 \geq 0\}$,

$\underline{g_U}(x_1,x_2) = \{(x_1,x_2) \in R^2 | -4x_1 + x_2 \leq 0, x_1 \geq 0\}$.$\underline{g_L}(x_1,x_2) \subseteq g_L(x_1,x_2) \subseteq \overline{g}_L(x_1,x_2)$,
$\overline{g}_L(x_1,x_2) = \{(x_1,x_2) \in R^2 | x_1 + x_2 \leq 60, x_2 \geq 0\}$,$\underline{g_L}(x_1,x_2) =$
$\{(x_1,x_2) \in R^2 |\ 4x_1 + x_2 \leq 50, x_2 \geq 0\}$.

The solution:
$CP_1$is solved, $A = \{(7.2,12.8)\}$, $A_1 = \{(7.2,12.8)\}$ which satifices the $\underline{g}_U(x_1,x_2)$& $\underline{g}_L(x_1,x_2)$ constrains so the optimun values are $f_U = 2304, f_L = 0$.

Example 4:

$$\min_{x_1,x_2} f_U(x_1,x_2,x_3,x_4) = (x_1-3)^2 + (x_3-4x_2+1)^2 + x_4$$
$$\text{S.T:}$$
$$(x_1,x_2,x_3,x_4) \in g_U(x_1,x_2,x_3,x_4)$$
$$\min_{x_3,x_4} f_L(x_1,x_2,x_3,x_4) = (x_1+2x_3)^2 + x_2 - 3x_4$$
$$\text{S.T}$$
$$(x_1,x_2,x_3,x_4) \in g_L(x_1,x_2,x_3,x_4)$$

where $\underline{g}_U(x_1,x_2,x_3,x_4) \subseteq g_U(x_1,x_2,x_3,x_4) \subseteq \overline{g}_U(x_1,x_2,x_3,x_4)$,

$$\overline{g}_U(x_1,x_2,x_3,x_4) = \left\{(x_1,x_2,x_3,x_4) \in R^4 \middle| \begin{array}{c} 2x_1 + x_2 + x_2^3 + x_4 - 6 \geq 0, \\ x_1 + 2x_2 + x_3 + x_4 - 30 \leq 0, x_1 \geq 0, x_2 \geq 0 \end{array}\right\},$$

$$\underline{g}_U(x_1,x_2,x_3,x_4) = \left\{(x_1,x_2,x_3,x_4) \in R^4 \middle| \begin{array}{c} 3x_1 - 2x_2 + x_2^3 - 2x_4 - 6 \geq 0, \\ 2x_1 + 3x_2 + x_3 + x_4 - 30 \leq 0, x_1 \geq 0, x_2 \geq 0 \end{array}\right\},$$

$$\underline{g}_L(x_1,x_2,x_3,x_4) \subseteq g_L(x_1,x_2,x_3,x_4) \subseteq \overline{g}_L(x_1,x_2,x_3,x_4),$$

$$\overline{g}_L(x_1,x_2,x_3,x_4) = \left\{(x_1,x_2,x_3,x_4) \in R^4 \middle| x_1 + \tfrac{1}{2}x_2^3 + 2x_3 + x_4^2 \leq 40, x_3 \geq 0, x_4 \geq 0\right\},$$

$$\underline{g}_L(x_1,x_2,x_3,x_4) = \{(x_1,x_2,x_3,x_4) \in R^4 | 2x_1 + x_2^3 + 4x_3 + x_4^2 \leq 40, x_3 \geq 0, x_4 \geq 0\}.$$

The solution:
we form problems CP_1, CP_2, CP_3 and CP_4 as table1. First solve CP_1 problem, the solution is $A = \{(3,0.25,0,6.08)\}, f_U = 7.08, f_L = -8.99, A_1 = \emptyset, A_2 = \emptyset, A_3 = \emptyset$. Solve problem CP_2, the solution set is $A_2 = \{(3,0.25,0,1.25)\}, f_U = 2.25, f_L = 5.5$. Solve problem CP_3, the solution set is $A_3 = \{(3,0.25,0,5.8296)\}, f_U = 6.8296, f_L = -8.2388$. The solution set of $CP_4 = \{(3,0.25,0,1.25)\}, f_U = 2.25, f_L = 5.5$. As we see in this example all problems are solved because there is no cooperation with DM. If there is any cooperaton with DM, the number of problem will be less than four problems.

4.2 RBNPP when roughness on the objective function only:

The second stuatition $RBNPP_o$ when roughness on the objective function and the constrians are deterministic. $RBNPP_o$ can be defined as following:

$$\min_x f_U(x,y)$$
$$\text{S.T:} \quad g_U(x,y) \leq 0 \tag{17}$$
$$\min_y f_L(x,y)$$
$$\text{S.T:} \quad g_L(x,y) \leq 0$$

where, $\underline{f_U}(x,y) \leq f_U(x,y) \leq \overline{f}_U(x,y)$,$\underline{f_L}(x,y) \leq f_L(x,y) \leq \overline{f}_L(x,y)$, and $\underline{f_U}(x,y), \overline{f}_U(x,y)$, $\underline{f_L}(x,y), \overline{f}_L(x,y)$, $g_U(x,y)$, and $g_L(x,y)$ are functions assumed to be at least first continuously differentiable functions.

This problem can be also divided into four problems as shown in table 2. The solution procedure for solving $RBNPP_o$is presented in flowchart 2.

Table 2. RBNPP when roughness on the objective function and the constrains are deterministic

FP_1	FP_2	FP_3	FP_4
$min_x\ \underline{f_U}(x,y)$ $S.T{:}g_U(x,y)\leq 0$	$\min_x \underline{f_U}(x,y)$ S.T:$g_U(x,y)\leq 0$	$min_x\ \overline{f}_U(x,y)$ $S.T{:}g_U(x,y)\leq 0$	$\min_x \overline{f}_U(x,y)$ S.T:$g_U(x,y)\leq 0$
$min_y\ \underline{f_L}(x,y)$ $S.T{:}g_L(x,y)\leq 0$	$\min_y \overline{f}_L(x,y)$ S.T:$g_L(x,y)\leq 0$	$min_y\ \underline{f_L}(x,y)$ $S.T{:}g_L(x,y)\leq 0$	$\min_y \overline{f}_L(x,y)$ S.T:$g_L(x,y)\leq 0$

Definition 5: *The surely optimal solution set for $RBNPP_o$* **.**The optimal solution set $B_1=\{(\overline{x},\overline{y})\in B|\overline{f}_U(\overline{x},\overline{y})=\underline{f_U}$ and $\overline{f_L}(\overline{x},\overline{y})=\underline{f_L}\}$ is the surely optimal solution set of the problem (4), if $B_1\neq\emptyset$.
Definition 6: *The surely optimal solution set to 1st level and possibly optimal solution to 2nd level for $RBNPP_o$* **.** The optimal solution set (B_3) of FP_3 is the surely optimal solution set to 1st level and possibly optimal solution to 2nd level if $B_3\neq\emptyset$.
Definition 7: *The possibly optimal solution set to 1st level and surely optimal solution to 2nd level for $RBNPP_o$* .B_2 is the possibly optimal solution set to 1st level and surely optimal solution to 2nd level if $B_2\neq\emptyset$ where B_2 is the optimal solution set of FP_2.
Definition 8: *The possibly optimal solution set for $RBNPP_o$*
$B_4\left(B_4=\{(\overline{x},\overline{y})\in B|\overline{f}_U(\overline{x},\overline{y})=\overline{f}_U,\overline{f}_U(\overline{x},\overline{y})\neq\underline{f_U},\overline{f_L}(\overline{x},\overline{y})=\overline{f_L},\overline{f_L}(\overline{x},\overline{y})\neq\underline{f_L}\}\right)$is the possibly optimal solution set to 1st level and possibly optimal solution to 2nd level.

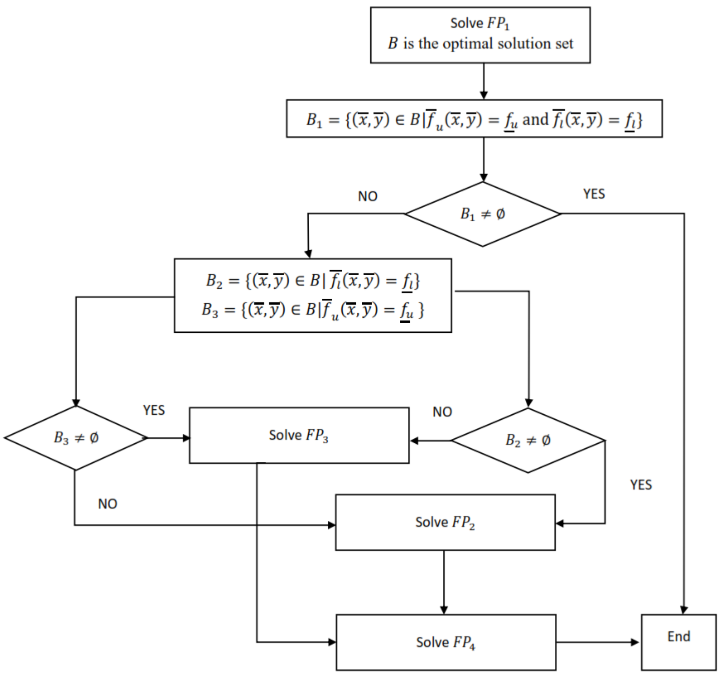

Figure 2. Flowchart 2 (The solution procedure for $RBNPP_o$)

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Example 5:

$$\min_{x_1,x_2} f_U(x_1,x_2,x_3,x_4)$$

S.T:

$$2x_1 + x_2 + x_2^3 + x_4 - 6 \geq 0,$$
$$x_1 + 2x_2 + x_3 + x_4 - 30 \leq 0,$$
$$x_1 \geq 0, x_2 \geq 0$$

$$\min_{x_3,x_4} f_L(x_1,x_2,x_3,x_4)$$

S.T

$$x_1 + \frac{1}{2}x_2^3 + 2x_3 + x_4^2 \leq 40,$$
$$x_3 \geq 0, x_4 \geq 0$$

where $\underline{f_U}(x_1,x_2,x_3,x_4) \leq f_U(x_1,x_2,x_3,x_4) \leq \overline{f}_U(x_1,x_2,x_3,x_4)$, $\overline{f}_U(x_1,x_2,x_3,x_4) = (x_1+1)^2 + (x_3+x_2)^2 + x_4$, $\underline{f_U}(x_1,x_2,x_3,x_4) = (x_1-3)^2 + (x_3-2x_2+1)^2 + x_4$, $\underline{f_L}(x_1,x_2,x_3,x_4) \leq f_L(x_1,x_2,x_3,x_4) \leq \overline{f}_L(x_1,x_2,x_3,x_4)$, $\overline{f}_L(x_1,x_2,x_3,x_4) = (2x_1+3x_3)^2 + x_2^3 + x_4$, $\underline{f_L}(x_1,x_2,x_3,x_4) = (x_1+2x_3)^2 + x_2 - 3x_4$.

The solution: We form problems FP_1, FP_2, FP_3 and FP_4 as table 1. First solve FP_1 problem, the solution is $B = \{(3,0.25,0,6.08)\}, f_U = 6.33, f_L = -8.73, B_1 = \emptyset, B_2 = \emptyset, B_3 = \emptyset$. Solve problem FP_2, the solution set is $B_2 = \{(3,0.25,0,6.08)\}, f_U = 6.33, f_L = -8.73$. Solve problem FP_3, the solution set is $B_3 = \{(3,0.25,0,6.08)\}, f_U = 6.33, f_L = -8.73$. The solution set of FP_4 is $\{(0,0,0,0)\}, f_U = 1, f_L = 0$.

4.3 RBNPP when roughness on the objective function and the constrians :

The third stuatition when rough on both the objective function level and the constrians. It can divided into sixteen problems as showen in table 3. The sutiable solution procedure here, solve problem P_1, C is the optimal solution set and $\underline{f_U}, \underline{f_L}$ are optimum values. Find $C_1 = \{(\overline{x},\overline{y}) \in C | \overline{f}_U(\overline{x},\overline{y}) = \underline{f_U}, \overline{f_L}(\overline{x},\overline{y}) = \underline{f_L}, (\overline{x},\overline{y}) \in \{\underline{g_U}(x,y) \cap \underline{g_L}(x,y)\}\}$. If $C_1 \neq \emptyset$, then is called the surely optimal solution set of the problem (1) which contains all surely optimal solution. If $C_1 = \emptyset$, find $C_2 = \{(\overline{x},\overline{y}) \in C | \overline{f}_U(\overline{x},\overline{y}) = \underline{f_U}, \overline{f_L}(\overline{x},\overline{y}) = \underline{f_L}, (\overline{x},\overline{y})\}$, $C_3 = C \cap \underline{g_U}(x,y) \cap \underline{g_L}(x,y)$. If $C_2 \neq \emptyset$, we will solve our problem as the procedure introduced in second situation. If $C_3 \neq \emptyset$, we will solve our problem as the procedure introduced in 1st situation. Or we can solve P_1 and P_{16} problems only because they give overall veiwe of the problem its optimu values varies.

Example 6:

$$\min_{x_1,x_2} f_U(x_1,x_2,x_3,x_4)$$

S.T:

$$(x_1,x_2,x_3,x_4) \in g_U(x_1,x_2,x_3,x_4)$$

$$\min_{x_3,x_4} f_L(x_1,x_2,x_3,x_4)$$

S.T

$$(x_1,x_2,x_3,x_4) \in g_L(x_1,x_2,x_3,x_4)$$

where $\underline{f_U}(x_1,x_2,x_3,x_4) \leq f_U(x_1,x_2,x_3,x_4) \leq \overline{f}_U(x_1,x_2,x_3,x_4)$,

$\overline{f}_U(x_1,x_2,x_3,x_4) = (x_1+x_4+1)^2 + (x_3+x_2)^2$, $\underline{f_U}(x_1,x_2,x_3,x_4) = (x_1-3)^2 + (x_3-2x_2+1)^2 + x_4$,
$\underline{g_U}(x_1,x_2,x_3,x_4) \subseteq g_U(x_1,x_2,x_3,x_4) \subseteq \overline{g}_U(x_1,x_2,x_3,x_4)$,$\overline{g}_U(x_1,x_2,x_3,x_4) =$
$\left\{(x_1,x_2,x_3,x_4) \in R^4 \middle| \begin{matrix} 3x_1-2x_2+x_2^3-4x_4+6 \geq 0, \\ x_1+x_2+x_3+x_4-30 \leq 0, x_1 \geq 0, x_2 \geq 0 \end{matrix}\right\}$,$\underline{g_U}(x_1,x_2,x_3,x_4) =$
$\left\{(x_1,x_2,x_3,x_4) \in R^4 \middle| \begin{matrix} 3x_1-2x_2+x_2^3-4x_4+6 \geq 0, \\ 2x_1+3x_2+x_3+x_4-30 \leq 0, x_1 \geq 0, x_2 \geq 0 \end{matrix}\right\}$,$\underline{f_L}(x_1,x_2,x_3,x_4) \subseteq$
$f_L(x_1,x_2,x_3,x_4) \subseteq \overline{f}_L(x_1,x_2,x_3,x_4)$,$\overline{f}_L(x_1,x_2,x_3,x_4) = x_2^3 + (2x_1+3x_3)^2 + x_4$,
$\underline{f_L}(x_1,x_2,x_3,x_4) = x_2 + (x_1+2x_3)^2 - 3x_4$,$\underline{g_L}(x_1,x_2,x_3,x_4) \subseteq g_L(x_1,x_2,x_3,x_4) \subseteq$
$\overline{g}_L(x_1,x_2,x_3,x_4)$,$\overline{g}_L(x_1,x_2,x_3,x_4) =$
$\{(x_1,x_2,x_3,x_4) \in R^4 | x_1+x_2^3+x_3+x_4^2 \leq 40, x_3 \geq 0, x_4 \geq 0\}$,$\underline{g_L}(x_1,x_2,x_3,x_4) =$
$\{(x_1,x_2,x_3,x_4) \in R^4 | 3x_1+x_2^3+4x_3+x_4^2 \leq 40, x_3 \geq 0, x_4 \geq 0\}$.

The solution:we solve only P_1 and P_{16} problems as we discussed before to take overall veiwe of the problem and its optimu values varies. If the DM sharing with the mathamtic the solution, he guied the mathamtic to how to solve the problem.

Conclusion:

This paper introduced an interactive approach for RBNPP solution. A new methodology is presented for solving BNPP taken into consideration which level is more important than the other. This methodology converted BNPP to a crisp unconstrained programming problem. The unconstrained programming problem is solved by a trust region penalty method. The solution procedures are discussed. The definitions of all types of solutions are presented.

The significant contributions of this paper:

- Introducing an interactive approach procedure between the mathematics and DM for solving RBNPP.
- Presenting an algorithm for solving BLP taken into consideration which level is more important than the other.
- Presenting the solution procedures of solving different types of roughness in BLP.
- The definitions of all types of solutions for different types of roughness in this problem are presented.
- For future work, this approach can be further expanded to treat the real problems using artificial intelligent algorithms or any techniques that handle the multilevel programming problems.

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