An Efficient Implementation of Coupling and Decoupling Scheme for Biharmonic Equation

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Abstract. A method for coupling and decoupling, utilizing finite difference, is developed to solve the biharmonic problem on a unit square. This problem is reformulated as a coupled system involving two second-order partial differential equations. This approach necessitates solving the original problem through a sequence of boundary value problems for the Poisson equation. It achieves this using a minimal number of mesh points, distinguishing itself from the traditional methods employed in prior research to address this particular issue. A compact finite difference scheme has been introduced for the solution of fourth and sixth-order Poisson equations. This innovative approach effectively reduces the computational cost of the proposed algorithm, especially when dealing with large grid numbers, compared to traditional methods. Simultaneously solving these Poisson equations can be easily programmed. We plan to apply this method to analyze the fourth-order differential problem of a square clamped plate subjected to various loads. The biharmonic problem has been examined with a focus on achieving higher-order accuracy. The outcomes of numerical experiments showcase the method's optimal global accuracy and reveal super convergence results. Notably, a sixth-order accuracy is observed at both the boundary nodes and interior points.

1. Introduction

The fourth-order differential equation holds significant importance in physics, particularly in the realms of fluid and solid mechanics. It has garnered considerable attention from scientists in the fields of engineering, mathematics, and computing sciences over the past few decades. Numerous iterative and non-iterative methods have been devised and implemented to address this problem, particularly when considering Dirichlet and/or Dirichlet and Neumann boundary conditions.

Various approaches exist, with some tackling the problem directly and solving the fourth-order equation in its original form. Here we choose a strategy involving the decomposition of the problem into two coupled Poisson equations. This approach leverages the simplicity and existing code for dealing with second-order equations, offering an alternative perspective in addressing the challenges posed by the fourth-order differential equation. This is the beginning of this approach of this special interpretation of decoupling and coupling [1].

Several numerical techniques have been explored for solving the biharmonic problem. Noteworthy among these are works based on the finite difference method, as published in [2], and a fast Poisson solver also rooted in the finite difference method (FDM) [3], [4]. Finite Line Method (FLM) is introduced as a solution technique for addressing both general linear and non-linear high-order partial differential equations [5]. These approaches typically entail higher-order differential approximations and involve a specific number of points in the mesh grid.

In the realm of finite element methods (FEM), there is the hp-version discontinuous Galerkin formulation [6], as well as a mixed finite volume method [7]. Lin et al. recently proposed an effective implementation of the weak Galerkin finite element methods for the biharmonic equation [8]. For more comprehensive details on FEM for solving the biharmonic equation, interested readers are directed to references in [9][10][11][12], as well as the citations therein.

Other methodologies include the Homotopy Analysis Method [13][14], the Fast Multipole Method [15], the Fast Fourier-Galerkin Method [16], and a multigrid technique proposed in [17]. These diverse methods contribute to the array of approaches available for approximating solutions to biharmonic problems.

The conventional finite difference method applied to biharmonic problems typically necessitates a substantial number of mesh points, especially when addressing higher-order differential equations (e.g., 13 points for solving the biharmonic equation) [18][19]. A strategy to mitigate this demand involves decomposing higher-order differential equations into several second-order differential equations. While solving these second-order equations simultaneously can be programmatically concise, it comes at the cost of increased computer time and memory requirements.

In this paper, we aim to develop a classical method for solving higher partial differential equations with varying degrees of accuracy through the application of a coupling technique. The core concept of this method involves breaking down the biharmonic equation into two decoupled Poisson equations, for which numerous efficient algorithms are available. Compact finite difference schemes are then employed to handle fourth and sixth-order accuracy for the biharmonic equation.

While the convergence of the mesh regrating process is theoretically established at the continuous level using operator equation theory, there exists a gap in the literature regarding numerical experiments to validate the method's effectiveness. Therefore, in this concise paper, we present the results of several numerical experiments with different levels of accuracy. This verification of convergence is crucial in confirming the method's reliability. The insights gained from these experiments not only contribute to the development of a fast and efficient algorithm for the biharmonic equation but also hold promise for tackling various higher partial differential equations through coupling and decoupling schemes.

The paper is structured as follows: In Section 2, we provide a concise introduction to the design of the coupling and decoupling scheme, outlining the application of compact finite difference for solving the biharmonic equation with higher-order accuracy. Section 3 is dedicated to presenting numerical results for biharmonic problems, serving as an application of plate theory with second, fourth, and sixth-order accuracy, respectively. At the end, we draw conclusions based on our findings.

2. A Coupling and Decoupling Method for Biharmonic Equation

In this section, for the sake of simplicity, we will focus on a reduced system of equations representing the fourth-order partial differential equation, specifically the biharmonic

equation. It is important to note that our results and methodologies can be extended to more complex and general systems of equations
Consider, for $w(x, y)$[20]:

$$\Delta^2 w = f(x, y) \ in\ \Omega, \quad w = 0 \ and\ \frac{\partial w}{\partial n} = 0 \ on\ \Gamma, \tag{1}$$

Posed on the unit square $\Omega = (0.1) \times (0.1)$ with specified boundary conditions. The notation $\frac{\partial \mathrm{w}}{\partial \mathrm{n}}$ refers to the normal derivative on the boundary of the unit square, where n is a unit vector orthogonal to the surface,

$$\frac{\partial \mathrm{w}}{\partial \mathrm{n}} = \begin{cases} \frac{\partial \mathrm{w}}{\partial \mathrm{x}} = 0, & 0 \leq x \leq 1 \\ \frac{\partial \mathrm{w}}{\partial \mathrm{y}} = 0, & 0 \leq y \leq 1 \end{cases} \tag{2}$$

Recall that,

$$\Delta^2 w = \frac{\partial^4 w}{\partial \mathrm{x}^4} + 2\frac{\partial^4 w}{\partial \mathrm{x}^2 \partial y^2} + \frac{\partial^4 w}{\partial \mathrm{y}^4} = f(x, y) \tag{3}$$

Previously, the biharmonic equation was addressed using the classical 13-point central difference formula [19]. In the current paper, our focus is on developing a classical method for solving high-order partial differential equations through the application of coupling and decoupling techniques. This novel method simplifies the fourth-order problem by breaking it down into two second-order problems, facilitating a more straightforward implementation.

Now the equation Eq (1) can be replaced by

$$\begin{gathered} \Delta w \equiv w_{xx} + w_{yy} = v, \\ \Delta v \equiv v_{xx} + v_{yy} = f(x, y) \end{gathered} \tag{4}$$

Considering the given boundary conditions Eq (2), our approach involves solving the finite difference analog of the coupled system represented by Eq (4) and Eq (2).

Superimpose a square grid over the unit square with size $h = \frac{1}{N} + 1$ for some positive integer N. let Ω be those gride points $(x, y) = (ih, jh)$ for $1 \leq i, j \leq N$ (i.e., the interior points), and let Γ be those boundary points. Let u be a function where $w(x, y) \equiv w_{ij}$.

A standard explicit finite difference scheme is employed to approximate the Poisson equation, utilizing a centered scheme as defined in Eq (5). It's noteworthy that the local truncation error of this approximation is of the order $O(h^2)$ [21].

$$\Delta w = \frac{w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - 4w_{i,j}}{h^2} \tag{5}$$

Enhanced approximations can be achieved by elevating the order of the truncation error in the finite difference scheme. This improvement is typically achieved by incorporating more points in the stencil of the numerical schemes, as demonstrated in Eq (6) and Eq (7). Notably, these schemes exhibit a local truncation error of $o(h^4)$ and $o(h^6)$, respectively [22][23].

$$\Delta w = \frac{-w_{i+2,j}+16w_{i+1,j}+16w_{i-1,j}-w_{i-2,j}-w_{i,j+2}+16w_{i,j+1}+16w_{i,j-1}-w_{i,j-2}-60w_{i,j}}{12h^2} \quad (6)$$

$$\Delta w = \frac{2w_{i+3,j}-27w_{i+2,j}+270w_{i+1,j}+270w_{i-1,j}-27w_{i-2,j}+2w_{i-3,j}+2w_{i,j+3}-27w_{i,j+2}+270w_{i,j+1}+270w_{i,j-1}-27w_{i,j-2}+2w_{i,j-3}-980w_{i,j}}{60h^2} \quad (7)$$

One drawback of this approach is the necessity to introduce additional equations for grid points, especially those in close proximity to and directly at the boundaries.

Compact finite difference (C.F.D) schemes offer the advantage of achieving higher-order accuracy with fewer cell points. These implicit methods, however, often involve matrix inversion when applied to solving partial differential equations. Specifically, second derivative compact finite difference schemes are employed for achieving fourth-order accuracy.

$$\frac{1}{10}w_{i-1}'' + w_i'' + \frac{1}{10}w_{i+1}'' = \frac{6}{5}\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$$

The proof for the second derivative compact finite difference (C.F.D) formula mentioned earlier can be found in [24]. Additionally, a rapid implementation of a fourth-order compact finite difference scheme for the Poisson equation in two dimensions is presented.

$$w_{i+1,j}^{xx} + 10w_{i,j}^{xx} + w_{i-1,j}^{xx} = \frac{12}{h_x^2}(w_{i+1,j} - 2w_{i,j} + w_{i-1,j})$$

$$w_{i+1,j}^{yy} + 10w_{i,j}^{yy} + w_{i-1,j}^{yy} = \frac{12}{h_y^2}(w_{i+1,j} - 2w_{i,j} + w_{i-1,j})$$

By means of Kronecker product (or tensor product) ⊗, one can reformulate above equations into the following matrix form

$$A_x \otimes I \;\; W^{xx} = \frac{12}{h_x^2} B_x \otimes I \;\; W$$

$$A_1 \;\; W^{xx} = \frac{12}{h_x^2} B_1 \, W$$

$$W^{xx} = \frac{12}{h_x^2} \, A_1^{-1} \, B_1 \, W$$

$$A_y \otimes I \;\; W^{yy} = \frac{12}{h_y^2} B_y \otimes I \;\; W$$

$$A_2 \;\; W^{yy} = \frac{12}{h_y^2} B_2 \, W$$

$$W^{yy} = \frac{12}{h_y^2} \, A_2^{-1} \, B_2 \, W$$

Where A_x, A_y andB_x, B_y taking the general form

$$A_x = \begin{pmatrix} 10 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 10 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 10 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 10 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 10 \end{pmatrix}, \quad B_x = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix}$$

$$\because \ \Delta w = W^{xx} + W^{yy} = F$$

$$\left(\frac{12}{h_x^2} A_1^{-1} B_1 + \frac{12}{h_y^2} A_2^{-1} B_2\right) \underline{W} = \underline{F}$$

If $h_x = h_y = h$,

$$\frac{12}{h^2} (A_1^{-1} B_1 + A_2^{-1} B_2) \underline{W} = \underline{F}$$

$$\underline{W} = \frac{h^2}{12} C^{-1} \underline{F}$$

Where,

$$C = A_1^{-1} B_1 + A_2^{-1} B_2$$

Similarly, the implementation of a six-order compact finite difference scheme for the Poisson equation in two dimensions is carried out in matrix form, as detailed in [25].

$$W_{xx} + W_{yy} = \left(\frac{3}{4h^2} A_1^{-1} B_1 + \frac{3}{4h^2} A_2^{-1} B_2\right) \underline{W}$$

$$\underline{W} = \frac{4h^2}{3} C^{-1} \underline{F}$$

Where,

$$C = A_1^{-1} B_1 + A_2^{-1} B_2$$

Where A_1, A_2 and B_1, B_2 taking the general form of $\tilde{A}$ and $\tilde{B}$ respectively:

$$\tilde{A} = \begin{pmatrix} 11 & 2 & 0 & .. & 0 & 0 \\ 2 & 12 & 2 & .. & 0 & 0 \\ 0 & 2 & 12 & .. & 0 & 0 \\ : & : & : & .. & : & : \\ : & : & : & .. & : & : \\ 0 & 0 & 0 & .. & 12 & 1 \\ 0 & 0 & 0 & .. & 1 & 12 \end{pmatrix},$$

$$\tilde{B}=\begin{pmatrix} -34 & 16 & 1 & 0 & 0 & .. & 0 & 0 & 0 \\ 16 & -34 & 16 & 1 & 0 & .. & 0 & 0 & 0 \\ 1 & 16 & -34 & 16 & 1 & .. & 0 & 0 & 0 \\ : & : & : & : & : & .. & : & : & : \\ : & : & : & : & : & .. & : & : & : \\ : & : & : & ; & : & .. & : & : & : \\ : & : & : & : & 1 & 16 & -34 & 16 & 1 \\ 0 & 0 & 0 & .. & 0 & 1 & 16 & -34 & 16 \\ 0 & 0 & 0 & .. & 0 & 0 & 1 & 16 & -34 \end{pmatrix}$$

We now propose to implement the decoupling method and utilize finite-difference techniques to obtain the matrix representation of the coupled system given by Eq (4).

$$\begin{aligned} \mathrm{A}\underline{W} &= \underline{V} \\ \mathrm{A}\underline{V} &= \underline{F} \end{aligned} \tag{8}$$

Combining the two expressions from Eq (8) results in a unified global matrix equation:

$$\mathrm{A}\left(\mathrm{A}\underline{W}\right)=\mathrm{A}^2\underline{W}=\underline{F} \tag{9}$$

Where, Matrix A for accuracy $O(h^2)$,

$$\mathrm{A}=\begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix}$$

Matrix A for fourth order compact finite difference $O(h^4)$,

$$\mathrm{A}=\begin{pmatrix} -0.4492 & 0.1237 & -0.0125 & .. & .. & .. & 0 \\ 0.1237 & -0.4492 & 0.1237 & .. & .. & : & : \\ -0.0125 & 0.1237 & -0.4492 & .. & .. & : & : \\ : & : & : & .. & : & : & : \\ : & : & : & .. & : & : & : \\ : & : & : & .. & -0.4492 & 0.1237 & -0.0125 \\ : & .. & .. & .. & 0.1237 & -0.4492 & 0.1237 \\ 0 & .. & .. & .. & -0.0125 & 0.1237 & -0.4492 \end{pmatrix}$$

Subsequently, it is necessary to apply the boundary conditions to matrix A^2 based on the appropriate accuracy level of the matrix. The prescribed boundary conditions for $O(h^2)$ accuracy are as follows:

$$\frac{\partial \mathrm{w}}{\partial \mathrm{n}} = \begin{cases} \dfrac{w_{0,j} - w_{1,j}}{2h}, & j = 1,2 \dots, N \\ \dfrac{w_{N+1,j} - w_{N,j}}{2h}, & j = 1,2 \dots, N \\ \dfrac{w_{i,0} - w_{i,1}}{2h}, & i = 1,2 \dots, N \\ \dfrac{w_{i,N+1} - w_{i,N}}{2h}, & i = 1,2 \dots, N \end{cases} \tag{10}$$

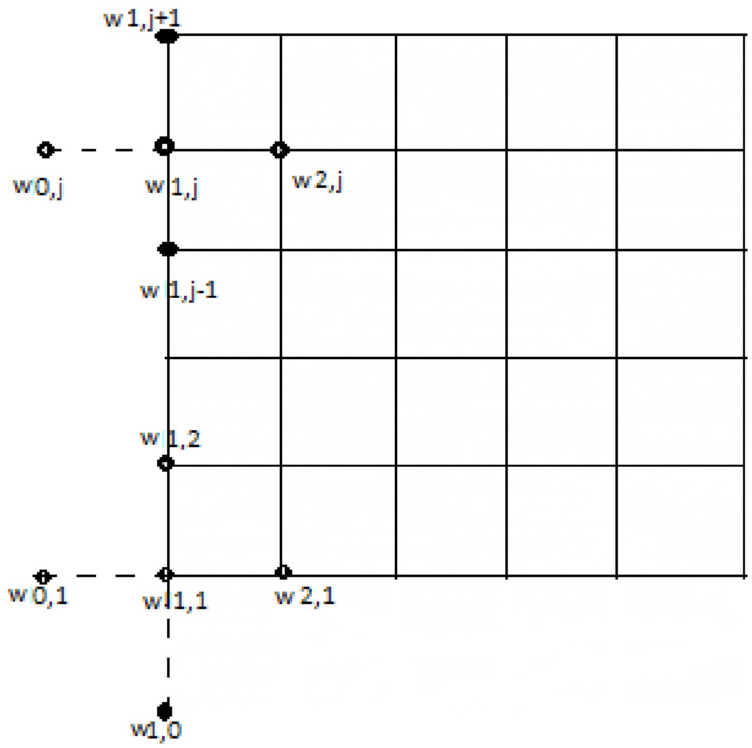

Figure 1. Ghost points for Neumann boundary conditions

At the boundary, we will deal the Neumann boundary condition, necessitating the inclusion of an additional row of unknowns outside the region (ghost points) [26]. For the Poisson equation with Neumann boundary conditions:

$$\Delta w = f(x, y) \;\; in\; \Omega, \quad \frac{\partial \mathrm{w}}{\partial \mathrm{n}} = g \; on \; \Gamma,$$

A natural approximation to the normal derivative involves employing a central difference scheme:

$$\frac{\partial w}{\partial n}(x_1, y_j) = \frac{w_{0,j} - w_{2,j}}{2h} + o(h^2)$$

To handle Neumann boundary conditions with increased accuracy, we introduce ghost points positioned outside of the domain and adjacent to the boundary.

The value $w_{0,j}$ is not well defined. We need to eliminate it from the equation. This is possible since on the boundary point (x_1, y_j), we have two equations;

$$-4w_{1,j} + w_{2,j} + w_{0,j} + w_{1,j+1} + w_{1,j-1} = h^2 f_{1,j} \tag{11}$$

$$w_{0,j} - w_{2,j} = 2hg_{1,j} \tag{12}$$

From Eq (12), we get $w_{0,j} = 2hg_{1,j} + w_{2,j}$. Substituting it into Eq (11), we get an equation at point (x_1, y_j);

$$-4w_{1,j} + 2w_{2,j} + w_{1,j+1} + w_{1,j-1} = h^2 f_{1,j} - 2hg_{1,j} \tag{13}$$

The scaling is implemented to preserve the symmetry of the matrix. We can address other boundary points using the same technique, with the exception of the four corner points.

At corner points, the normal vector is not well-defined. To approximate, we utilize the average of two directional derivatives. Using the example of (0,0), we have:

$$-4w_{1,1} + w_{2,1} + w_{0,1} + w_{1,2} + w_{1,0} = h^2 f_{1,1}, \tag{14}$$

$$w_{0,1} - w_{2,1} = 2hg_{1,1}, \tag{15}$$

$$w_{1,0} - w_{1,2} = 2hg_{1,1}, \tag{16}$$

So, we can solve $w_{0,1}$and $w_{1,0}$ from Eq (15) and Eq (16), and substitute them into Eq (14). This gives an equation for the corner point (x_1, y_1),

$$-4w_{1,1} + 2w_{2,1} + 2w_{1,2} = h^2 f_{1,1} - 2hg_{1,j}, \tag{17}$$

Similar techniques will be employed to handle other corner points. In the same manner, we apply the boundary conditions with higher-order accuracy to matrices.

The entire numerical procedure can be summarized as follows:

1. Given i, j, M, N and grid points $(x_i, y_j), i = 0,1, \dots, M, j = 0,1, \dots, N$, compute $\underline{F}$ by evaluation of $f(x_i, y_j)$ h.
2. Compute A and A^2 defined in Eq. (9).
3. Apply the boundary conditions on A^2.
4. Perform the matrix inversion of A^2 and get W.

3. Numerical Realization of The Coupling Method

We have developed a decoupling and coupling scheme for the biharmonic equation with second, fourth, and six-order accuracy, respectively. In this section, we will use the proposed numerical algorithm to compute the deflection of a clamped plate under various loads and assess the convergence of the process. The examples below consider a computational domain represented by a rectangle covered with a uniform grid. All computations are performed on a laptop computer with an Intel(R) Core (TM) i7-5500U CPU, utilizing MATLAB.

Example 4.1

$$\Delta^2 w = 1 \; in \; \Omega, \quad w = 0 \; and \; \frac{\partial \mathrm{w}}{\partial \mathrm{n}} = 0 \; on \; \Gamma,$$

This problem corresponds to the bending of a square clamped plate under a uniform load [27]. Table 1 provides the computed values of the deflection at point $(0.5, 0.5)$. It is worth noting that, in comparison, [27] reports values $of\ w(0.5, 0.5) \approx 0.001265$. Our results, achieved with second-order accuracy, are evidently comparable to those reported in [27].

Table 1 Rate of convergence for bending of clamped plate under uniform load of the different accuracy

Step size	Second order $\boldsymbol{o(h^2)}$	Fourth order $\boldsymbol{o(h^4)}$	Sixth order $\boldsymbol{o(h^6)}$
0.12500	0.00048103	0.000675769	0.000893484
0.06250	0.00075309	0.000919317	0.001121958
0.04545	0.00095365	0.001020987	0.001203811
0.03125	0.00105619	0.001124602	0.001244686
0.01851	0.00115018	0.001190624	0.001265651
0.01562	0.00118642	0.001232904	0.001265485
0.00925	0.00121763	0.001249588	0.001265369
0.00463	0.00124076	0.001254223	0.001265366

Table 2 Comparison of error between the of different accuracy in case of Example 4.1

Step size	0.125	0.0625	0.045455	0.03125	0.018519	0.015625	0.009259	0.00463
error								
2nd order	7.8397e-04	5.1190e-04	3.1134e-04	2.0881e-04	1.1481e-04	7.8572e-05	4.7364e-05	2.4234e-05
4th order	5.0150e-04	3.4568e-04	2.4401e-04	1.4040e-04	7.4376e-05	3.2096e-05	1.5412e-05	1.0777e-05
6th order	2.9689e-4	1.4304e-4	6.1189e-05	2.0314e-05	6.5149e-07	4.85e-07	3.6930e-07	3.6622e-07

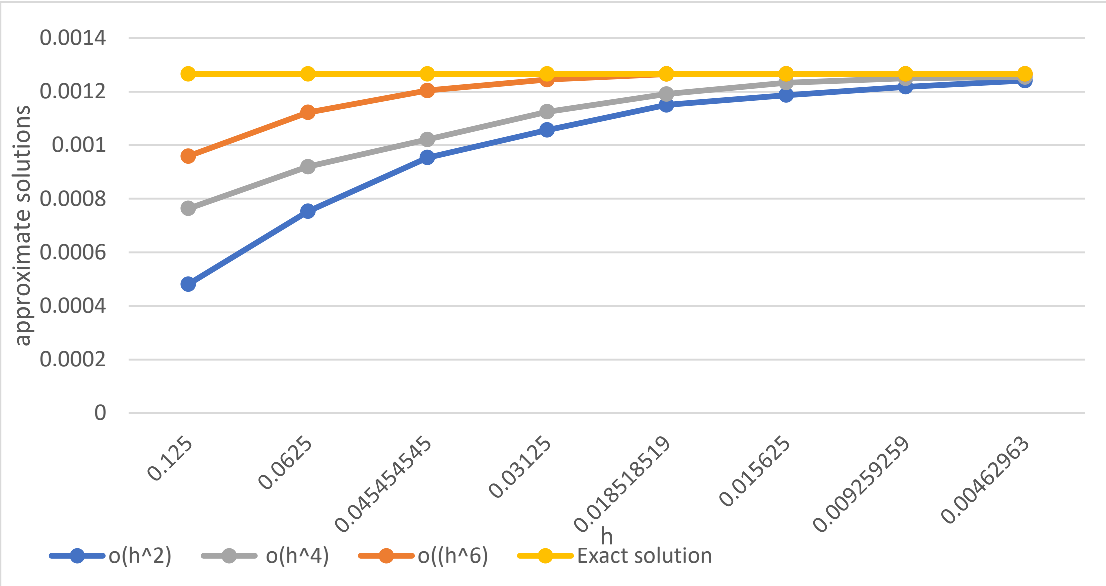

Figure 2. Comparison of solution of different accuracy Example 4.1

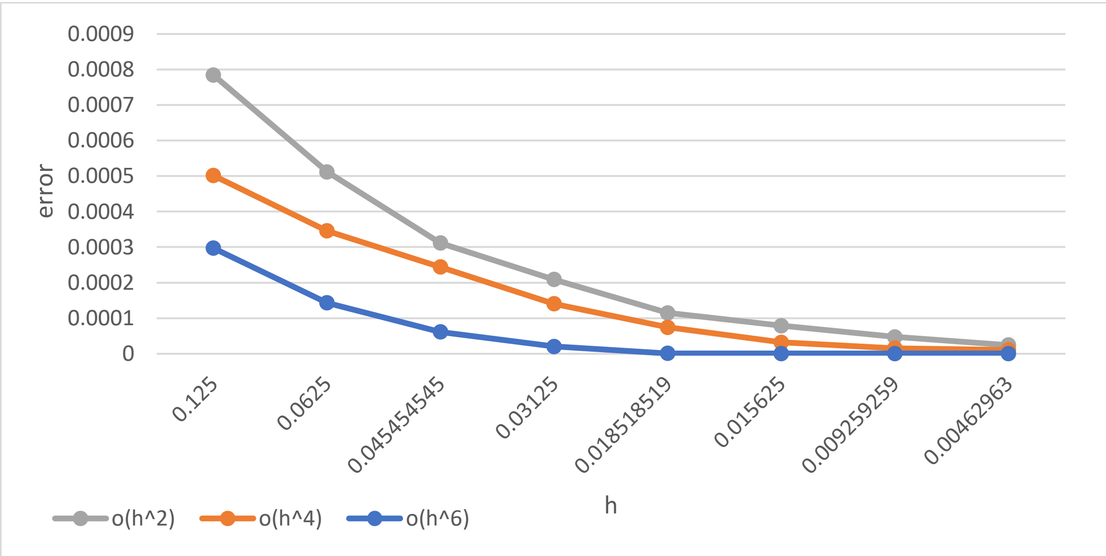

Figure 3. Plots of ordinary error with different mesh sizes for different accuracy for Example 4.1.

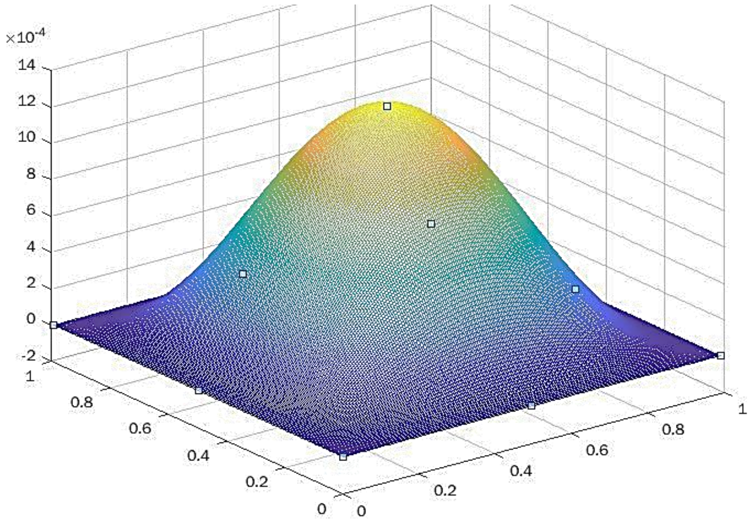

Figure 4. Solution plot of the Example 4.1 for M = 217.

Example 4.2

$$\Delta^2 w = x \quad in\ \Omega, \quad w = 0\ and\ \frac{\partial \mathrm{w}}{\partial \mathrm{n}} = 0\ on\ \Gamma,$$

This particular problem represents the bending of a square clamped plate under a linear load [28]. The computed values of the deflection at point $(0.5, 0.5)$ are presented in Table 3. In [28], corresponding values are reported as $w(0.5, 0.5) \approx 0.00063$. Once again, we achieve results that are comparable to those reported in [28].

Table 3 Rate of convergence of the bending of clamped plate under linear load in different accuracy

Step size	Second order $\boldsymbol{o}(\boldsymbol{h}^2)$	Fourth order $\boldsymbol{o}(\boldsymbol{h}^4)$	Sixth order $\boldsymbol{o}(\boldsymbol{h}^6)$
0.12500	0.000240516	0.000276353	0.000294743
0.06250	0.000436298	0.00044354	0.000479556
0.04545	0.000476829	0.000510494	0.000549793
0.03125	0.000518711	0.000552586	0.000593104
0.01851	0.000554027	0.000572512	0.000624702
0.01562	0.000585934	0.000593214	0.000631826
0.00925	0.000600111	0.000603525	0.000631569
0.00463	0.000616035	0.000608818	0.000631061

Table 4 Comparison of error between the of different accuracy in case of Example 4.2

Step size	0.125	0.0625	0.045455	0.03125	0.018519	0.015625	0.009259	0.00463
Error								
2nd order	3.8948e-4	1.9370e-04	1.5317e-4	1.1129e-4	7.5973e-05	4.4066e-05	2.9889e-05	1.3965e-05
4th order	3.5365e-4	1.8646e-04	1.1951e-04	7.7414e-05	5.7488e-05	3.6786e-05	2.6475e-05	2.1182e-05
6th order	3.3526e-04	1.5044e-04	8.0207e-05	3.6896e-05	5.2978e-06	1.8257e-06	1.5686e-06	1.0609e-06

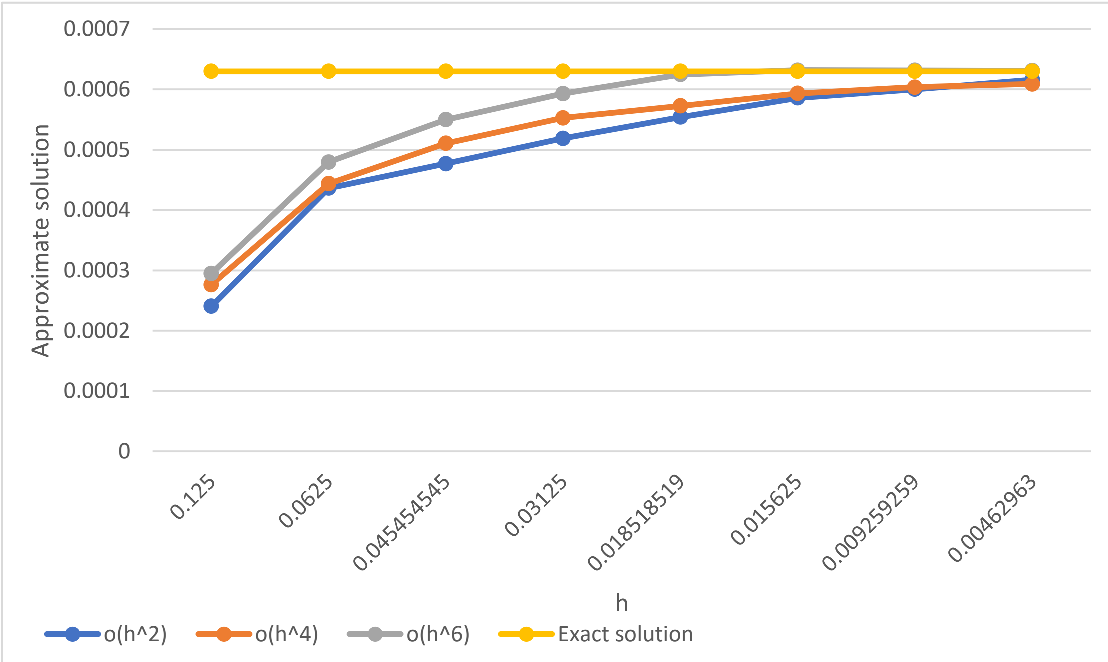

Figure 5. Comparison of solution of different accuracy Example 4.2

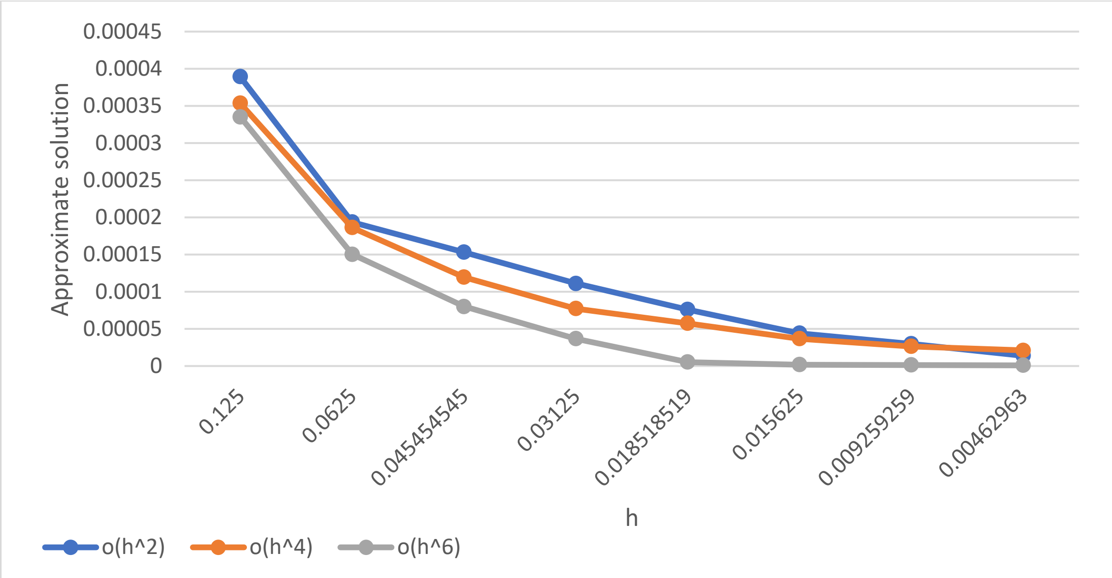

Figure 6. Plots of ordinary error with different mesh sizes for different accuracy for Example 4.2.

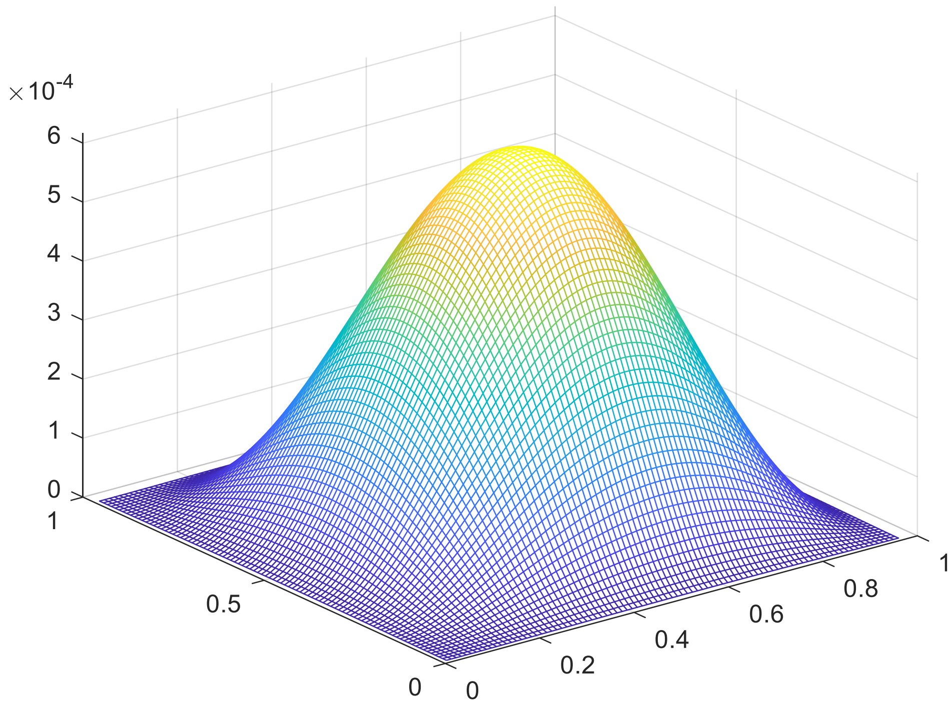

Figure 7. Solution plot of the Example 4.2 for M = 217.

Example 4.3

$$w(x,y) = x^3(1-x)^3y^3(1-y)^3,$$

This particular problem corresponds to the Chinosi problem, where a unit square plate is considered with uniform decomposition into triangular elements, and all edges are built-in [29]. The value of the deflection at point $(0.5, 0.5) = 0.000244140625$ under the specified load

$$\begin{aligned} f(x,y) = & -72(1-x)^2y^3(1-y)^3 + 216x(1-x)y^3(1-y)^3 - 72\,x^2\,y^3(1-y)^3 \\ & + 72x(1-x)^3y(1-y)^3 - 216x(1-x)^3y^2(1-y)^2 \\ & + 72x(1-x)^3y^3(1-y) - 216x^2(1-x)^2y(1-y)^3 \\ & + 648x^2(1-x)^2y^2(1-y)^2 - 216x^2(1-x)^2y^3(1-y) \\ & + 72x^3(1-x)y(1-y)^3 - 216x^3(1-x)y^2(1-y)^2 \\ & + 72x^3(1-x)y^3(1-y) - 72x^3(1-x)^3(1-y)^2 \\ & + 216x^3(1-x)^3y(1-y) - 72x^3(1-x)^3y^2 \end{aligned}$$

Table 5. Rate of convergence of the chinosi problem with different accuracy

Step size	Second order $\boldsymbol{o(h^2)}$	Fourth order $\boldsymbol{o(h^4)}$	Sixth order $\boldsymbol{o(h^6)}$
0.12500	0.000234713	0.000239398	0.000245918
0.06250	0.000241364	0.000240088	0.000245111
0.04545	0.000245525	0.000242006	0.000244728
0.03125	0.000244924	0.000243499	0.000244531
0.01851	0.000244635	0.000244815	0.000244418
0.01562	0.000244596	0.000244436	0.000244253
0.00925	0.000244479	0.00024443	0.000244215
0.00463	0.000244386	0.000244311	0.000244144

Table 6. Comparison of error between the of different accuracy in case of Example 4.3

Step size	0.125	0.0625	0.045455	0.03125	0.018519	0.015625	0.009259	0.00463
Error								
2nd order	9.43e-06	2.78e-06	1.38e-06	7.84e-07	4.95e-07	4.56e-07	3.39e-07	2.46e-07
4th order	4.7419e-06	4.0525e-06	2.1341e-06	6.4148e-07	6.7467e-07	2.9609e-07	2.8985e-07	1.7122e-07
6th order	1.7777e-06	9.7076e-07	5.8786e-07	3.9078e-07	2.7777e-07	1.1308e-07	7.5352e-08	3.5586e-09

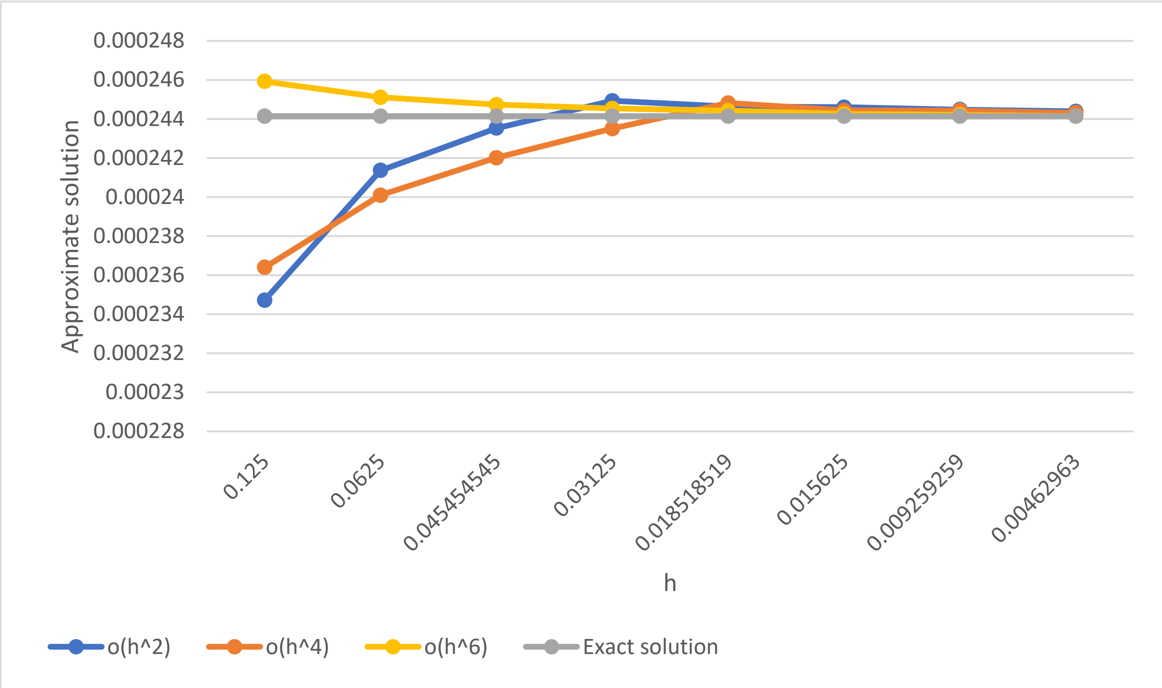

Figure 8. Comparison of solution of different accuracy Example 4.3

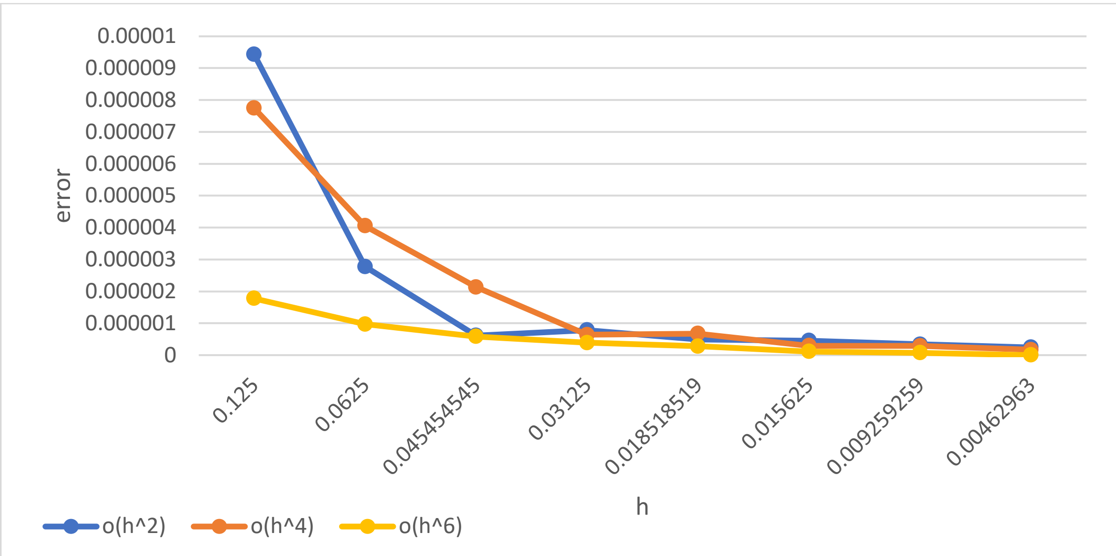

Figure 9. Plots of ordinary error with different mesh sizes for different accuracy for Example 4.3.

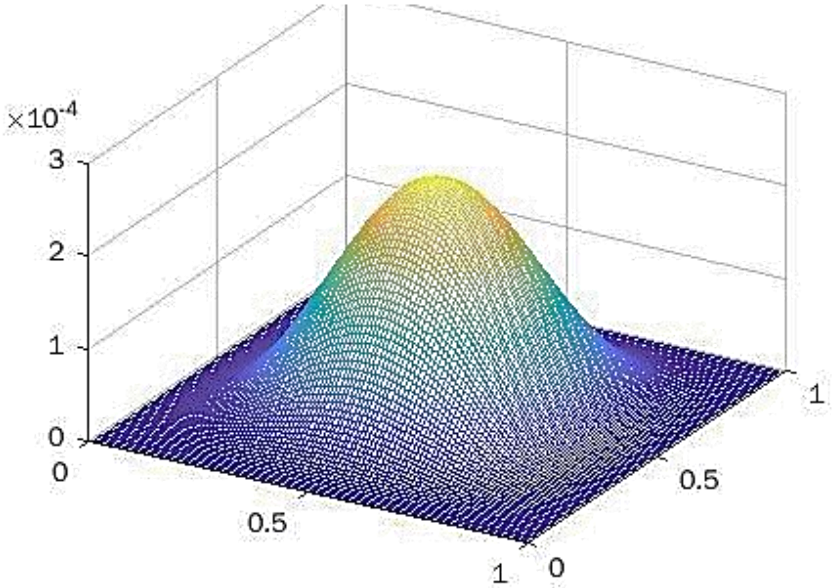

Figure 10. Solution plot of the Example 4.3 for M = 217.

Example 4.4

$$w(x,y) = (1-\cos(\pi x))^2(1-\cos(\pi y))^2$$

This problem models the bending of a square clamped plate under distributed load is described in the following form,

$$\begin{aligned} f(x,y) = {} & 8\,\pi^{\,4}\,cos(\pi x)^2(cos(\pi y)^2 - 1) + 8\,\pi^{\,4}\,cos(\pi y\,)^2(cos(\pi x)^2 - 1) \\ & - 8\,\pi^{\,4} sin(\pi x)^2(cos(\pi y)^2 - 1) - 8\,\pi^{\,4}\,sin(\pi y)^2(cos(\pi x)^2 - 1) \\ & + 7.89568352 * 10\ \pi^{\,2}\,cos(\pi x)^2\,cos(\pi y)^2 - 7.89568352 \\ & * 10\ \pi^{\,2}\,cos(\pi x)^2 sin(\pi y)^2 - 7.89568352 * 10\,\pi^{\,2}\,cos(\pi y)^2 sin(\pi x)^2 \\ & + 7.89568352 * 10\,\pi^{\,2}\,\sin(\pi x)^2 sin(\pi y)^2 \end{aligned}$$

The computed values of the deflection $w(0.5, 0.5)$ is represented in Table 7. The exact solution $w(0.5, 0.5) = 1$. we again obtain results of different accuracy

Table 7 Rate of convergence of the different accuracy in case of Example 4.4

Step size	Second order $\boldsymbol{o(h^2)}$	Fourth order $\boldsymbol{o(h^4)}$	Sixth order $\boldsymbol{o(h^6)}$
0.12500	0.661260577	0.698334435	0.726405113
0.06250	0.809711609	0.858964118	0.893533303
0.04545	0.885677195	0.903637915	0.942763069
0.03125	0.914088288	0.932841374	0.970844666
0.01851	0.932779584	0.964580534	0.988511616
0.01562	0.955003453	0.974689032	0.988829783
0.00925	0.972183561	0.983160346	0.998714487
0.00463	0.987882879	0.991281627	0.99890543

Table 8. Comparison of error between the of different accuracy in case of Example 4.4

Step size	0.125	0.0625	0.045455	0.03125	0.018519	0.015625	0.009259	0.00463
Error								
2^{nd}order	0.338739	0.190288	0.114323	0.085912	0.06722	0.044997	0.027816	0.012117
4^{th} order	0.301666	0.141036	0.096362	0.067159	0.035419	0.025311	0.01684	0.008718
6^{th}order	0.273595	0.106467	0.057237	0.029155	0.011488	0.01117	0.001286	0.001095

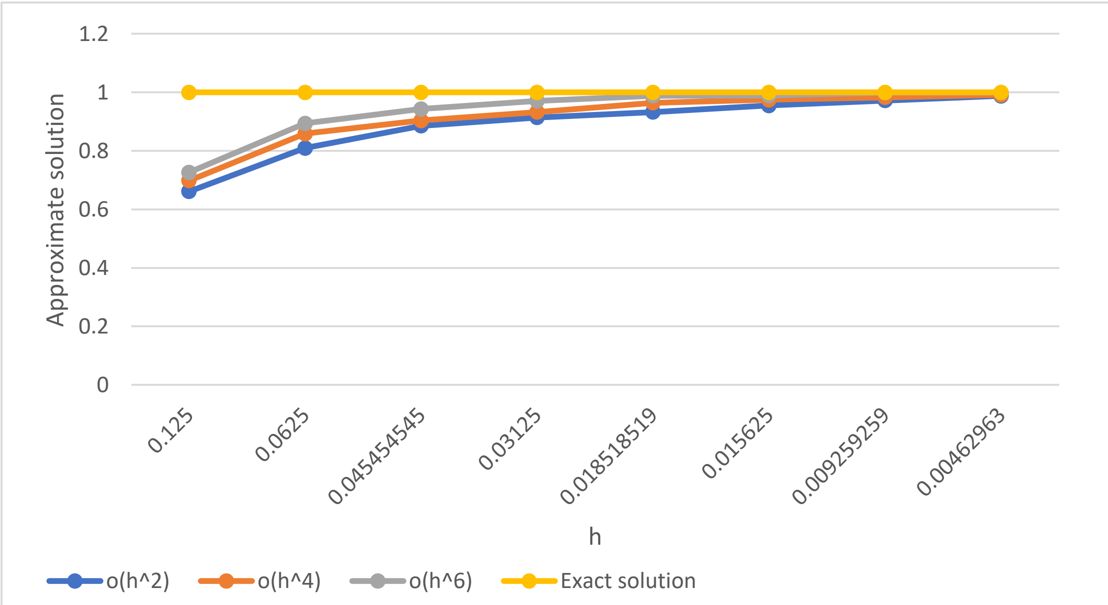

Figure 11. Comparison of solution of different accuracy Example 4.4

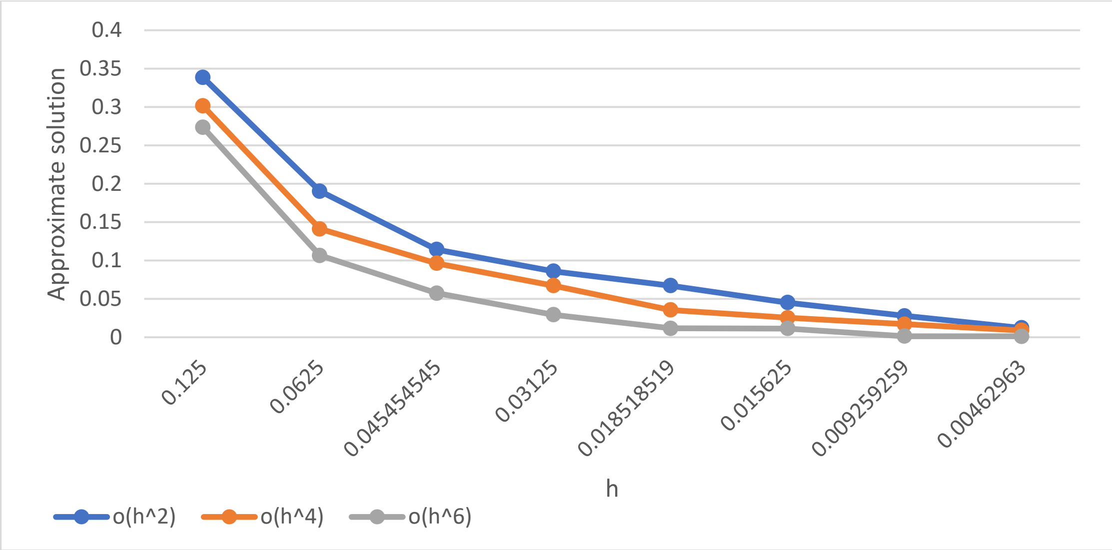

Figure 12. Plots of ordinary error with different mesh sizes for different accuracy for Example 4.4.

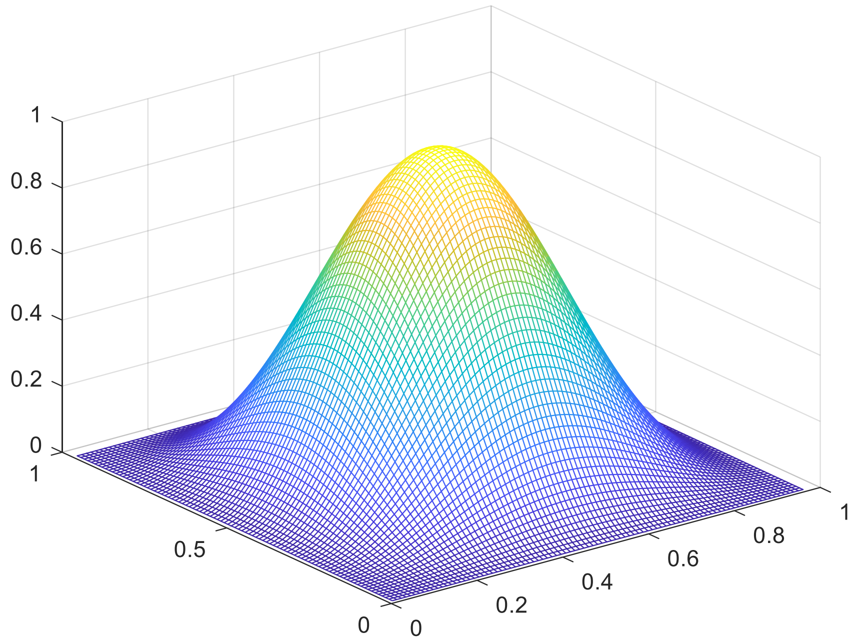

Figure 13. Solution plot of the Example 4.4 for M = 217.

4. Concluding Remarks

In this paper, we have introduced a method for solving the biharmonic equation within the unit square Ω, showcasing its optimality across various accuracy levels and demonstrating super convergence results. The presented numerical experiments highlight the rapid convergence of the coupling and decoupling method, particularly when employing the accuracy of $O(h^6)$. This observation underscores the effectiveness of the approach to reduce biharmonic problems to a sequence of second-order problems, requiring fewer grid points compared to classical methods. The utilization of compact finite difference schemes has proven instrumental in minimizing the grid points for solving the biharmonic equation with higher-order accuracy. This technique relies on a work on Laplace operator and use it for all powers of Laplace (Bi-Harmonic, Tri-Harmonic, Quad-Harmonic…). There will be no need for complicated and difficult to obtain stencils of poly-Hamonic operators.

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